PREDICTION OF THE LENGTH OF THE AIR CORE IN VORTICES OF WEAK CIRCULATION

BY
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1. Introduction

When a pipe draws water from a reservoir in the vicinity of solid boundaries it frequently happens that the normal pressure on the boundaries produces circulation in the water. The circulation is conserved except in a thin "viscous core," where viscous forces are of the same order of magnitude as inertia forces and where the motion changes from a potential vortex to solid rotation. When the viscous core extends from the pipe entrance to the free surface, the free surface is depressed to form an air core. The present paper is concerned with the prediction of the length of the air core when the circulation in the reservoir and the rate of withdrawal of water are specified, and the product $\Phi^2 R$ is low. (see Section 3 for notation).

Specifically the model studied is that shown in Figure 1 where the circulation at infinite radius (Γ_∞) and the axial mainstream velocity distribution (w_1) outside of the viscous core are known. Since only the length of the air core rather than the shape of the free surface is of concern, it will be sufficient to relate the pressure distribution in the viscous core below the air core to the variables w_{1P} and Γ_∞ .

The only comprehensive theoretical treatment of this problem which has been encountered is that of Einstein and Li, 1955. They assumed an axial velocity profile:

$$w = w_0 \quad r < \frac{d}{2}$$

$$w = 0 \quad r > \frac{d}{2}$$

and also that:

$$w_0 = w_{1P} \cdot \frac{z}{z_c} \quad z < z_c$$

1/ Geometry of theoretical model.
Géométrie du modèle théorique.

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2/ Axial velocity profiles.
Profils des vitesses axiales.

By this means the axial and tangential equations of motion were uncoupled and a simple formulation for the tangential velocity distribution (and hence pressure distribution) in the viscous core achieved. In the theory presented here a "profile" method of solution is again used but a rather more realistic profile for w is assumed and no assumption is made regarding the axial distribution of w .

In order to choose a suitable axial velocity profile it is essential to examine the structure of the viscous core observed experimentally.

2. Structure of the viscous core

In section 5.1 an apparatus is described which generates the boundary conditions of Figure 1. Experiments carried out in this apparatus indicated that the velocity profile in the viscous core was always of the type shown by curve "a" of Figure 2. This is demonstrated in Figure 3 which shows the distortion of a drop of dye at intervals after its insertion near the top of the air core. The fluid close to the axis is clearly moving downwards more slowly than that at larger radii. This result is not surprising. Lewellen, 1962, showed that as:

$$\Phi^2 \rightarrow \infty$$

$$w \rightarrow zf(r)$$

Therefore, since the lowest part of the air core is a stagnation point, $f(r) = 0$ when $r = 0$ and so w must be small along the whole of the axis of rotation. However, it was observed in the present experiments that as the product $\Phi^2 R$ was reduced, so the air core became thin, Figure 4, and the central region of low axial velocity occupied a decreasing proportion of the region of downwards axial velocity. Also, away from the air core, the velocity on the axis became of the same order of magnitude as that of the surrounding fluid. Therefore, the axial velocity profile chosen for the theoretical model is:

$$w - w_1 = (w_0 - w_1) \exp - \frac{r^2}{\delta_4^2} \quad (1)$$

3/ Distortion of a drop of dye placed near the top of the air core. Photographs taken at 0.15 sec. intervals $\Phi^2 R = 700$.
Distorsion d'une goutte de colorant introduite près du sommet du noyau d'air. Clichés pris à 0,15 s $\Phi^2 R = 700$.

4/ Typical air core.
Noyau d'air type.
 $F = 9.78$
 $\Phi^2 R = 55$

curve "b" Figure 2, on the understanding that $\Phi^2 \mathcal{R}$ must be small. The upper limit of $\Phi^2 \mathcal{R}$ is not specified here. In a following section simplified equations for the viscous core are derived and it is found that there is a maximum value of $\Phi^2 \mathcal{R}$ above which realistic solutions of these equations are not obtainable.

3. Notation

- r, z co-ordinates (see Figure 1);
 d suction pipe diameter;
 u radial velocity component;
 v tangential velocity component;
 w axial velocity component;
 w_1 axial mainstream velocity component (= axial velocity on the axis of symmetry in potential flow);
 w_{1P} axial mainstream velocity component in the suction pipe (= axial velocity on the axis of symmetry of the pipe in potential flow);
 Γ circulation vr ;
 p defined by $(p_s/\rho) - gz$ where ρ is the density of the fluid and p_s the static pressure above atmospheric;
 δ small quantity such that $\delta \ll d$;
 δ_4 "axial curvature thickness" defined by:

$$\frac{1}{\delta_4^2} = \frac{-1}{w_0 - w_1} \left[\frac{\partial w}{\partial r^2} \right]_0;$$

- ψ Stokes's stream function defined by:

$$w = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad u = -\frac{1}{r} \frac{\partial \psi}{\partial z};$$

- g gravitational constant;
 ν kinematic viscosity.

DIMENSIONLESS QUANTITIES:

$$Z = \frac{z}{d} \quad W = \frac{w}{w_{1P}} \quad W_1 = \frac{w_1}{w_{1P}}$$

$$\Delta^2 = \frac{\delta_4^2}{d^2} \mathcal{R}$$

$$\mathcal{R} = \frac{w_{1P} d}{\nu} \text{ (Reynolds number)}$$

$$\Phi = \frac{\Gamma_\infty}{w_{1P} d}$$

$$\mathcal{F} = \frac{w_{1P}}{\sqrt{gd}} \text{ (Froude number)}$$

SUBSCRIPTS:

- ₀ at the axis $r=0$;
 _{∞} at $r=\infty$;
_{cr} at the critical point.

4. Theory

4.1. EQUATIONS OF MOTION FOR THE VISCUS CORE:

We consider the system shown in Figure 1. The simplified equations of motion for the viscous core will be derived on the assumption that the free sur-

face remains plane, since this is of some intrinsic interest, and the solutions of these equations will be used directly to predict the length of the air core.

The complete equations of motion for steady axisymmetric incompressible viscous flow are:

$$\frac{u \partial u}{\partial r} + \frac{w \partial u}{\partial z} - \frac{v^2}{r} + \frac{\partial p}{\partial r} - \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \right] = 0$$

$$\frac{u \partial v}{\partial r} + \frac{w \partial v}{\partial z} + \frac{uv}{r} - \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} \right] = 0$$

$$\frac{u \partial w}{\partial r} + \frac{w \partial w}{\partial z} + \frac{\partial p}{\partial z} - \nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] = 0$$

and the boundary conditions of the present theoretical model are:

$$\begin{aligned} vr &= \Gamma_\infty & r &= \infty \\ w &= 0 & z &= 0 \end{aligned}$$

also we have:

$$\frac{d}{dz} \int_0^\infty (p + w^2) r dr = 0$$

Two assumptions simplify the above equations: Assumption (i) viscous effects are confined within a core of small radius $0(\delta)$.

In the viscous core the orders of magnitude are:

$$\begin{aligned} r &\sim 0(\delta) & z &\sim 0(1) \\ w &\sim 0(1) & u &\sim 0(\delta) \\ v &\sim 0(1) & \nu &\sim 0(\delta^2) \end{aligned}$$

[where $d \sim z_c \sim 0(1)$]

Neglecting terms of the order δ and less, the equations of motion for the viscous core become:—

$$\frac{\partial p}{\partial r} = \frac{v^2}{r} \quad (2)$$

$$\frac{u \partial v}{\partial r} + \frac{w \partial v}{\partial z} + \frac{uv}{r} - \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) = 0 \quad (3)$$

$$\frac{u \partial w}{\partial r} + \frac{w \partial w}{\partial z} + \frac{\partial p}{\partial z} - \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) = 0 \quad (4)$$

Assumption (ii) The axial mainstream velocity w_1 outside of the viscous core is the same as the potential flow velocity along the axis of symmetry.

The problem is now to solve equations 2, 3, 4 for the core with the mainstream velocity w_1 and the circulation Γ_∞ specified. The boundary conditions of the simplified equations are therefore:—

$$\begin{aligned} w &= w_1 & r &= \infty \\ vr &= \Gamma_\infty \\ w &= 0 & z &= 0 \end{aligned}$$

4.2. INTEGRATION OF THE AXIAL VELOCITY EQUATION:

Following the usual procedure with profile methods of solution equation 4 is integrated along the radius and the resulting integrals expressed in terms of w_0 and δ_4 by use of equation 1. Although equation 4 is simplified by this process, an additional equation relating δ_4 to conditions at $r=0$ is introduced.

Integrating equation 4 between limits $r=0$ and $r=h$ where $h \gg 0$ (δ) lies in the region of potential flow, and making use of the continuity equation:

$$\frac{\partial}{\partial r} ur + \frac{\partial}{\partial z} wr = 0$$

there results:

$$\frac{d}{dz} \int_0^h w^2 r dr - w_1 \frac{d}{dz} \int_0^h wr dr + \int_0^h \frac{\partial p}{\partial z} r dr = 0 \quad (5)$$

Now at $r=h$;

$$\frac{\partial p}{\partial z} = -w_1 \frac{dw_1}{dz}$$

Therefore, when $r < h$, from equation 2:

$$\begin{aligned} \frac{\partial p}{\partial z} &= -w_1 \frac{dw_1}{dz} - \frac{\partial}{\partial z} \int_r^h \frac{\Gamma^2}{r^3} dr \\ &= -w_1 \frac{dw_1}{dz} - \frac{\partial}{\partial z} \int_r^\infty \frac{\Gamma^2}{r^3} dr \end{aligned}$$

Substituting for $\partial p / \partial z$ in equation 5 and expressing the integrals involving w in terms of w_0 and δ_4 by means of equations we get:

$$\begin{aligned} &\left(\frac{w_0^2 - w_1^2}{4} \right) \frac{d\delta_4^2}{dz} + \\ &+ \delta_4^2 \left(w_0 \frac{dw_0}{dz} + w_0 \frac{dw_1}{dz} - w_1 \frac{dw_1}{dz} \right) - \\ &- \int_0^\infty \frac{\partial}{\partial z} \int_r^\infty \frac{\Gamma^2}{r^3} dr dr = 0 \quad (6) \end{aligned}$$

A second equation involving w_0 and δ_4 is derived by observing the conditions to be fulfilled at $r=0$. Here we have, from equation 4:

$$\frac{\partial p}{\partial z} + w_0 \frac{dw_0}{dz} - \nu \frac{\partial^2 w}{\partial r^2} = 0$$

Again, using equations 1 and 2:

$$\begin{aligned} 2 \frac{(w_0 - w_1)}{\delta_4^2} &= \\ &= -\frac{1}{\nu} \left(w_0 \frac{dw_0}{dz} - w_1 \frac{dw_1}{dz} + \frac{d}{dz} \int_0^\infty \frac{\Gamma^2}{r^3} dr \right) \quad (7) \end{aligned}$$

There are now two equations involving w_0 and δ_4 , viz. equations 6 and 7, and it remains to express the derivatives involving circulation in terms of w_0 and δ_4 by means of equation 3.

4.3. SOLUTION OF EQUATION 3:

The axial velocity profile may be written:

$$w - w_1 = (w_0 - w_1) - (w_0 - w_1) \frac{r^2}{\delta_4^2} + \dots$$

A solution to equation 3 will be obtained using only the leading term in the expansion. This procedure is akin to that used by Lighthill 1950 in connection with the solution of the temperature equation in a laminar boundary layer on a heated surface.

In terms of circulation Γ , equation 3 is:

$$\frac{u \partial \Gamma}{\partial r} + \frac{w \partial \Gamma}{\partial z} - \nu \left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) = 0 \quad (8)$$

Transforming from co-ordinates r and z to co-ordinates ψ and z , equation 8 is:

$$\frac{\partial \Gamma}{\partial z} = \nu \left(r \frac{\partial wr}{\partial \psi} \frac{\partial \Gamma}{\partial \psi} - \frac{\partial \Gamma}{\partial \psi} \right)$$

and since for $w = w_0$, $\psi = (w_0 r^2 / 2)$, then:

$$\frac{\partial \Gamma}{\partial z} = 2 \nu \psi \frac{\partial^2 \Gamma}{\partial \psi^2}$$

Now since $\Gamma = 0$ when $\psi = z = 0$ a further transformation:

$$\eta = \frac{2 \psi}{z \nu}$$

is permissible. The last equation is now:

$$\frac{d^2 \Gamma}{d\eta^2} + \frac{1}{4} \frac{d\Gamma}{d\eta} = 0$$

whose solution, taking into account the boundary conditions:

$$\Gamma = 0 \quad \eta = 0$$

$$\Gamma = \Gamma_\infty \quad \eta = \infty$$

is:

$$\frac{\Gamma}{\Gamma_\infty} = 1 - e^{-\eta/4} = 1 - e^{-\frac{w_0 r^2}{4 z \nu}} \quad (9)$$

A means of obtaining a solution to equation 3 which takes account of the curvature term in the series for w is given in Woodford 1962.

The terms involving Γ in equations 6 and 7 may now be simply expressed in terms of w_0 , for we have, using equation 9:

$$\frac{d}{dz} \int_0^\infty \frac{\Gamma^2}{r^3} dr = \frac{\Gamma_\infty^2}{4 \nu} \ln 2 \frac{d}{dz} \frac{w_0}{z} \quad (10)$$

and:

$$\int_0^\infty \frac{\partial}{\partial z} \int_r^\infty \frac{\Gamma^2}{r^3} dr dr = -\frac{\Gamma_\infty^2}{4} \frac{d}{dz} \ln \frac{w_0}{z} \quad (11)$$

4.4. THE SIMPLIFIED EQUATIONS OF MOTION FOR THE VISCOUS CORE AND THEIR SOLUTION:

We may now substitute the right hand side of equations 10 and 11 for the integrals involving circulation in equations 6 and 7. It will be convenient to express the simplified equations of motion

so obtained in dimensionless form. Carrying out the substitution and rearranging we get:

$$\frac{d\Delta^2}{dZ} = \frac{1}{W_0^2 - W_1^2} \left\{ -4 \left(\frac{W_0 \Delta^2}{2} - \frac{\Phi^2 \mathcal{R}}{4 W_0} \right) \frac{dW_0}{dZ} \right\} \quad (12)$$

$$\frac{dw_0}{dz} = \frac{-\frac{2}{\Delta^2} (W_0 - W_1) + \frac{W_1 dW_1}{dZ} - \frac{\Phi^2 \mathcal{R} W_0 \ln 2}{4 Z^2}}{W_0 - \frac{\Phi^2 \mathcal{R} \ln 2}{4 Z}} \quad (13)$$

which must be solved simultaneously. Some observations on the nature of W_1 will aid their solution. In the model of Figure 1 the axial velocity W_1 is large enough to be significant in the above equations only in the vicinity of the suction pipe entrance, $Z > Z_b$ (see also Figure 5); it is negligibly small for $Z < Z_b$. Accordingly, for small Z , let $W_1 = 0$. Series solutions for W_0 and Δ^2 are now feasible. With:

$$W_0 = a_1 Z + a_2 Z^2 + a_3 Z^3 + \dots$$

we find that choice of:

$$a_1 = G = \left(\frac{dW_0}{dz} \right)_{z=0}$$

is arbitrary and the series:

$$W_0 = GZ + 4.24 \frac{G^2 Z^3}{\Phi^2 \mathcal{R}} + \dots$$

$$\Delta = \frac{4.18}{G} + \dots \quad (14)$$

result after evaluating the remaining coefficients.

The value of G is found in the following way. For small Z the leading term in the denominator of equation 13 is $-(0.173 \Phi^2 \mathcal{R}/Z)$ so that the denominator is negative initially. At large Z :

$$W_0 \rightarrow W_1$$

$$\frac{0.173 \Phi^2 \mathcal{R}}{Z} \rightarrow 0$$

so the leading term is W_1 which is positive. Therefore the denominator is zero at some value of Z . To maintain dW_0/dZ finite throughout Z the numerator and denominator of equation 13 must become zero simultaneously. G is chosen fulfil this condition.

The method of solution of the simplified equations of motion is therefore this: W_0 and Δ^2 are calculated at $Z = Z_b$ from equations 14 for a guessed value of G . Using these values as initial conditions equations 12 and 13 are solved numerically by a marching process in the region $Z_b < Z < Z_c$ until either the numerator or denominator of equation 13 changes sign. Values of G are adjusted so that ultimately the numerator and denominator are zero simultaneously.

The significance of the point $Z = Z_{cr}$, where dW_0/dZ is indeterminate, is that flow in the core for $Z < Z_{cr}$ is uninfluenced by the behaviour of the

mainstream at $Z > Z_{cr}$. The point Z_{cr} will subsequently be called a critical point.

It should be observed that realistic solutions of equations 12 and 13 are possible only if $\Phi^2 \mathcal{R}$ is less than some maximum. For the only positive term in the numerator of equation 13 is $W_1 dW_1/dZ$, so that Z_{cr} , where the numerator is zero, must lie within $Z_b < Z < Z_c$ where dW_1/dZ is of the same order of magnitude as other terms in the equation. From the denominator of equation 13:

$$W_0 \sim \Phi^2 \mathcal{R}$$

so that, at the critical point, the numerator gives:

$$\frac{W_1 dW_1}{dZ} \sim (\Phi^2 \mathcal{R})^2$$

But $W_1 dW_1/dZ$ has a fixed maximum value in the interval $Z_b < Z < Z_c$. Hence, if $\Phi^2 \mathcal{R}$ becomes too large, no physically real solution of the equations exists.

4.5. DERIVATION OF THE EQUATION FOR THE LENGTH OF THE AIR CORE:

The simplified equations of motion and their solutions in the interval $Z < Z_b$, viz equation 14 will now be used to predict the length of an air core when the bottom of the core lies within $Z < Z_b$. We make the assumption that equation 14 apply within $Z_a < Z < Z_b$, that is, the relationship between W_0 and Δ^2 is unaffected by deflection of the free surface from the plane state.

At the bottom of the air core:

$$p_\infty - p_0 = \int_0^\infty \frac{\Gamma^2}{r^3} dr = gz$$

Substituting for the integral from equation 10:

$$\frac{\Gamma_\infty^2}{4 \nu} \ln 2 \cdot \frac{W_0}{z} = gz$$

which is, in dimensionless form:

$$0.176 \Phi^2 \mathcal{R} W_0 - \frac{Z^2}{\mathcal{F}^2} = 0$$

Inserting the series of equation 14 for W_0 and rearranging we get:

$$Z_a = \frac{0.669}{G^2 \mathcal{F}^2} [1 - \sqrt{1 - 0.526 \Phi^2 \mathcal{R} \mathcal{F}^4 G^3}] \quad (15)$$

Where the positive root is neglected. This is the required formula. Note that G is a function of $\Phi^2 \mathcal{R}$, and for specified $\Phi^2 \mathcal{R}$ may be found in the manner described in the previous section. Thus:

$$Z_a = f(\mathcal{F}, \Phi^2 \mathcal{R}) \quad (16)$$

4.6. A NUMERICAL EXAMPLE FOR A DEPTH OF WATER $Z_c = 14.5$.

The equations of motion 12 and 13, have been solved for a particular depth of water $Z_c = 14.5$. The results serve to justify the order of magnitude assumption in section 4.1 and form a basis of comparison with experimental results.

5/ Mainstream axial velocity distribution.
Répartition des vitesses axiales. Courant principal.

6/ Relationship between G and $\phi^2 R$ for $Z_c = 14.5$.
Variation de G en fonction de $\phi^2 R$, pour $Z_c = 14,5$.

7/ Relationship between theoretical air core length Z_a and $\phi^2 R$ for various Froude Nos.
Variation de la longueur théorique du noyau d'air Z_a en fonction de $\phi^2 R$ pour diverses valeurs du nombre de Froude.

The mainstream velocity distribution for $Z_b < Z < Z_c$ is shown in Figure 5. This was derived from a relaxation solution for Stokes's stream function using a square net side 0.2. Using this distribution of W_1 , equations 12 and 13 were solved on an IBM 7090 computer in the way suggested in section 4.4. It was not possible to satisfy the 0/0 condition perfectly since, for a particular value of $\Phi^2 R$, only a finite number of values of G could be tried. The computer selected two adjacent values of G from a table of descending values of G, one of which was too large and the other too small. These pairs of values of G are shown on the graph of Figure 6 and it appears that, to the order of accuracy worked to, the relationship between G and $\Phi^2 R$ is a linear one. Using these results equation 15 was solved for Z_a for a range of Froude numbers $2.73 \leq Fr \leq 9.78$. In Figure 7 values of Z_a thus obtained are plotted with $\Phi^2 R$ as abscissa for various Froude numbers. An alternative arrangement in Figure 12 relates Fr to $\Phi^2 R$ for $(Z_a/Z_c) = 0.25, 0.50$ and 0.75 . The slope of the lines plotted here is exactly -1 so that, for Z_a/Z_c constant

$$Fr \Phi^2 R = \text{constant}$$

which means that, for the particular value of $Z_c = 14.5$ chosen here, the length of the air core is independent of the suction pipe velocity W_{1p} . From equation 15 it may be deduced that this is a consequence of the linearity in the relationship between $\Phi^2 R$ and G.

8/ Axial velocity W_0 and curvature thickness $\Delta^2 \sim Z$ for a plane free surface $\phi^2 R = 100$.
Vitesse axiale W_0 et épaisseur de courbure $\Delta^2 \sim Z$, pour une surface libre et plane $\phi^2 R = 100$.

Typical solutions for W_0 and Δ^2 are shown in Figure 8. The discontinuities in the curves at Z_b are due to errors in the series approximations at this point which arise because of the small number of terms in the series.

In order to satisfy the order of magnitude assumption of section 4.1 we should have:

$$\Delta = \frac{\delta \sqrt{R}}{d} \sim 0(\delta) \cdot 0(1/\delta) \sim 0(1)$$

For the particular example illustrated $\Delta \approx 7.5$ which is rather high. This value is lowered if the submergence of the suction pipe is decreased. The order of magnitude assumptions will not apply, then, if the submergence Z_c is much greater than about 15, (which is in accordance with $d \sim z_c \sim 0(1)$ of Section 4.1).

5. Experiment

The aim of the experiments was to measure air core lengths for a range of values of $\Phi^2 R$ and $\mathcal{F}$ and a depth of water $Z_c = 14.5$. These measurements were then directly comparable with the theoretical results of the preceding section. Two other groups of measurements were taken under geometrically similar conditions but with different depths of water, the idea being to see whether or not the dependence of Z_b upon $\mathcal{F}$ and $\Phi^2 R$ was affected by a change in R alone.

5.1. APPARATUS:

The apparatus was designed to reproduce as nearly as possible the ideal boundary conditions of the theoretical model.

A matt black honeycomb drum (actuator drum) was suspended from a thrust bearing in a square, glass sided tank, Figure 9. Water entered the tank through four perforated vertical pipes set in the corners and passed first through stilling screens and then through the actuator drum before leaving the tank through a 3 in. bore vertical suction pipe on the axis of symmetry at the bottom of the tank. A long 3 in. bore pipe containing a British Standard flow measuring orifice and a flow control valve connected the suction pipe to a centrifugal pump which returned the water to the corners of the tank to complete a closed circuit hydraulic system, Figure 10.

The actuator drum was revolved by means of a governor controlled variable speed electric motor. Water passing through the drum was given a circulation independent of the flow rate. The speed of the drum, and hence the circulation of the water, was measured by timing a single revolution of the drum with a stop watch.

The flow rate in the suction pipe was measured by means of an air-water and a water-mercury manometer connected in parallel across the B.S. orifice using the d and $d/2$ system of tappings. The air-water manometer could be isolated for high flow readings.

The mirrors and camera shown in Figure 9 were not used for the measurement of the air core described in the next section.

5.2. GENERATION AND MEASUREMENT OF THE AIR CORE:

For the experiments with $Z_c = 14.5$ the tank was filled to its near maximum depth (21.75 in.) and the 3 in. bore suction pipe reduced to 1 1/2 in. at its mouth by inserting a diffuser with a short 1 1/2 in. bore parallel entrance. In this way it was possible to obtain suction velocities of up to 24 ft/s.

According to equation 16:

$$Z_a = f(\Phi^2 R \mathcal{F})$$

It was therefore convenient to conduct experiments with the suction pipe velocity constant (and therefore $\mathcal{F}$ constant) and the circulation (and therefore $\Phi^2 R$) varied because in this way experimental and theoretical results were directly comparable.

The lengths of the air cores were measured by means of the two transparent scales affixed to

9 a/ Actuator drum.
Cylindre générateur du vortex.

9 b/ Vortex tank.
Cube à vortex.

10/ Hydraulic circuit.
Schéma du circuit hydraulique.

II a/

II b/

II c/

II d/

II e/

II a/ Air core length z_a vs. $\phi^2 R$. $F = 9.78$.
Longueur du noyau d'air z_a en fonction de $\phi^2 R$
($F = 9,78$).

II b/ Air core length z_a vs. $\phi^2 R$. $F = 7.68$.
Longueur du noyau d'air z_a en fonction de $\phi^2 R$
($F = 7,68$).

II c/ Air core length z_a vs. $\phi^2 R$. $F = 6.27$.
Longueur du noyau d'air z_a en fonction de $\phi^2 R$
($F = 6,27$).

II d/ Air core length z_a vs. $\phi^2 R$. $F = 4.59$.
Longueur du noyau d'air z_a en fonction de $\phi^2 R$
($F = 4,59$).

II e/ Air core length z_a vs. $\phi^2 R$. $F = 2.73$.
Longueur du noyau d'air z_a en fonction de $\phi^2 R$
($F = 2,73$).

opposite sides of the tank, Figure 10. The air core was viewed through one of the scales and the level of the eye adjusted until the bottom of the air core and two identical readings on the scales became coincident. The scale reading thus taken marked the position of the bottom of the air core. The length of the core fluctuated appreciably regardless of how long the suction velocity and circulation had been constant. Therefore several readings of core length were taken over a period of about 5 mn and it is the means of these results that are plotted on the graphs of Figure 11.

In Figure 12 $\mathcal{F}$ and $\Phi^2 R$ are related for three air core lengths ($Z_a/Z_c = 0.25, 0.5$ and 0.75). The experimental points on this graph were obtained by interpolation from Figure 11.

Two further series of experiments were conducted in the manner outlined above. In the first, a depth of water 22" and a 3" dia. suction pipe were used, and in the second a depth of 11" and a 1 1/2" dia. suction pipe. The two experiments were therefore conducted under geometrically similar conditions. By arranging for equality of $\mathcal{F}$ and $\Phi^2 R$ in the two experiments, the effect of the change in scale (or Reynolds number) was demonstrated. The results appear in Figure 13 which was obtained in a way similar to Figure 12.

6. Discussion of the results

In Figure 12, a graph relating $\mathcal{F}$ and $\Phi^2 R$, the slope of both experimental and theoretical lines is -1 . According to section 4.6 we may conclude from this that the length of the air core is independent of circulation within the range of $\Phi^2 R$ considered here. The theoretical result is a consequence of the large submergence of the suction pipe chosen, i.e. $Z_c \gg Z_c - Z_b$. We may arrive at this conclusion as follows. Since the submergence is large:

$$Z_{cr} \approx Z_b \quad W_{cr} \approx W_b$$

Therefore, from the denominator of equation 13:

$$W_b \approx \frac{0.173 \Phi^2 R}{Z_b} \propto \Phi^2 R$$

and from the series solution for W_0 we have, using the above result:

$$W_b = GZ_b + 4.24 \frac{G^2 Z_b^3}{\Phi^2 R} \propto \Phi^2 R$$

which is satisfied if $G \propto \Phi^2 R$. It has already been mentioned in section 4.6 that this is the relationship which gives rise to:

$$\mathcal{F} \Phi^2 R = \text{constant}$$

that is, the slope of -1 in Figure 12.

For a particular suction pipe velocity, the ratio of theoretical to experimental circulation which will produce an air core of given length is about 0.7. A principal reason for this error is that the effect of curvature of the axial velocity profile on the solution to the circulation equation 8 has been neglected. To account for this effect it would be necessary to solve a third differential equation

concurrently with equations 12 and 13 which calculated the ratio of the axial curvature thickness δ_4 to the tangential curvature thickness Δ_4 defined by:

$$\frac{1}{\Delta_4^2} = \frac{1}{\Gamma_\infty} \left(\frac{\partial \Gamma}{\partial r^2} \right)_0$$

Although greater accuracy would have been obtained by inclusion of this third equation, the salient features of the motion already described (which appear to be qualitatively correct) would not have changed.

The results taken from two geometrically similar

12/ F vs. $\Phi^2 R$ for air core lengths $Z_a/Z_c = 0.25, 0.5, 0.75$.
Variation du nombre de Froude en fonction de $\Phi^2 R$ pour des longueurs du noyau d'air $Z_a/Z_c = 0.25, 0.5, 0.75$.

13/ Variation of F with $\Phi^2 R$ for air core lengths $Z_a/Z_c = 0.25, 0.75$.
Variation du nombre de Froude en fonction de $\Phi^2 R$ pour des longueurs du noyau d'air $Z_a/Z_c = 0.25, 0.75$.

configurations with a scale ratio of 2 are shown in Figure 13. The correlation of results appears reasonable up to $\Phi^2 R \approx 150$. For $\Phi^2 R > 150$ the character of the motion changes completely and is such that, for $\mathcal{F}$ constant, the length of the air core is not sensitive to increase in $\Phi^2 R$. It was observed that $\Phi^2 R > 150$ corresponded with rapid growth of the central region of low axial velocity in the viscous core.

The characteristics of the flow are most clearly illustrated if simple physical parameters are used rather than dimensionless groups. Figure 14 shows a qualitative graph relating air core length z_a to circulation Γ_∞ for various suction pipe velocities (c.f. Figure 11). This is typical of the behaviour of the flow observed in experiments although only the results of the portion AB of the graph, to which the present theory applies, have been given in detail. The inclined line AB marks the phase where z_a is independent of w_{1p} . The horizontal lines demonstrate that as the circulation is increased, the air core length becomes independent of Γ but sensitive to changes in w_{1p} . It is this region which corresponds to the domain $\Phi^2 R > 150$ in Figure 13 where the central region of low axial velocity in the viscous core occupies a large proportion of the total core diameter.

It is clear that a theory which aims at a quantitative prediction of air core length for an arbitrary circulation, rather than that confined within the limit AB of Figure 14, must have implicit in it an axial velocity distribution of type "a", Figure 2 in the viscous core. The overestimates of air core length calculated by Einstein and Li, which were attributed to the neglect of turbulent effects in their theory, were more likely due to an erroneous choice of axial velocity profile.

7. Conclusions

A theory has been derived for predicting the length of the air core in a steady viscous laminar vortex when the product $\Phi^2 R$ is low. Within the limited range of experimental data presented the theory has been shown to be qualitatively correct. An interesting feature of the motion is that for

14/ Qualitative graph of $\log z_a$ vs. $\log \Gamma_\infty$.
Schéma qualitatif pour $\log z_a$ en fonction de $\log \Gamma_\infty$.

fairly large submergence of the suction pipe, ($Z_c \approx 10$ say), the length of the air core is independent of the suction pipe velocity.

When the product $\Phi^2 R$ is high the characteristics of the motion change completely. The air core length is almost independent of Γ_∞ and the central region of low axial velocity in the viscous core occupies a large proportion of the core. The present theory is not applicable to this type of flow.

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Résumé

Le calcul de la longueur du noyau d'air à l'intérieur des vortex à faible recirculation par D.J. Woodford *

Le modèle théorique considéré est représenté sur la figure 1. Des simples équations de mouvement seront formulées pour le noyau d'air, en admettant une surface libre plane, et les solutions de ces équations seront employées pour calculer la longueur du noyau d'air.

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