ANALYSIS OF SCOUR IN OPEN CHANNELS BY MEANS OF MATHEMATICAL MODELS

BY
M. H. GRADOWCZYK *
AND
H. C. FOLGUERA *

1. — Introduction

The classical problem of the scour in water streams and the evolution in time of the bed of open water courses are discussed in this paper from a new point of view.

Up to now these studies have been made mainly with the help of physical models and measurements *in situ*. It is well known that these models are sensible to error due to difficulties in finding an adequate scaling factor which may conciliate reliability with the physical aspects of the bed-load transport problem.

This paper aims to present a new approach by means of a mathematical model for the bed-load transport. A preliminar version of this model has been given by Gradowczyk [1].

To formulate the model, the different physical laws which are related to the phenomenon are vinculated in such a way that the mathematical model "simulates" the physical problem. Hence the model, which can be defined for deterministic or stochastic variables, reproduces the principal aspects of the scour in open channels and rivers.

This model provides also an efficient tool to check different kind of theoretical and empirical formulas for bed entrainement in steady and unsteady flow, with the advantage that the operation time and cost are respectively shorter and lower than the ones which correspond to physical models.

This new technique may be used not only as an independent method of analysis but also in combination with an adequate type of physical experimentation. It can also be extended as to consider the material moving in suspension in the fluid.

* Instituto de Cálculo- Facultad de Ciencias Exactas y Naturales de Buenos Aires.

2. — Definition of a mathematical model

Let us consider a physical problem which can be expressed in the following mathematical form:

$$\mathrm{L}\,(\mathrm{V}) = f\,(x, y, z, t), \qquad (1)$$

$$\mathrm{U}_j\,(\mathrm{V}) = g_j\,(x_0, y_0, z_0, t_0)\,; \qquad (j = 1,2, \ldots, m)$$

where $\mathrm{V}\,(x, y, z, t)$ is the unknown (or unknown vector), L is a differential operator of the n-th order, U_j are operators which indicate the boundary conditions of the problem, $f\,(x, y, z, t)$ is a known function and $g_j\,(x_0, y_0, z_0, t_0)$ are known functions at the boundaries.

If the operators L or U_j are non-linear operators, the system (1) is said to be non-linear. In general, no close solutions of the problem may exist. Besides the given formulation, an alternative formulation of a physical problem in terms of integral or functional operators can also be given.

In order to obtain a solution of system (1), approximate methods should be used, e.g. those which are suitable for digital computers.

When dealing with general non-linear systems, the difficulties encountered in the application of approximate methods, e.g. finite difference techniques, increase due to the non-linearity of the resulting algebraic equations. This happens even for simple problems (1). We may recall, on this behalf, that the Navier-Stokes equations had been successfully integrated by the finite difference method for fluids with a very low Reynolds number, as it was shown by Dimitrescu et *al.* [2].

This situation is rapidly changing due to the use of high speed digital computers with large capacity, as shown recently by Harlow and Fromm [3], who

(1) See for instance : S. H. Crandall « Engineering analysis », New York, 1956, p. 389.

discussed the flow of viscous incompressible fluids around an obstacle by means of the finite difference method using the complete equations for Reynolds numbers $\mathcal{R} < 6\,000$.

A different approach can also be introduced. Instead of studying the solution of system (1), we may reformulate the physical problem considering a simplified model. This simplified model should allow us to establish a new system of relations between the different unknown variables (elements of vector V), which may replace the original formulation of the problem as given by Eqs (1).

This system of relations can be evaluated numerically in a sequential form with a lesser degree of difficulties than the former system (1).

The system of relations which result from the simplified model is what we called "mathematical model" of the physical problem.

To establish a mathematical model, it is necessary to introduce a number of additional hypotheses in the physical problem. For example, in a problem of Engineering Hydraulics, it should be necessary to adopt simplified assumptions regarding hydrodynamic aspects, e.g. distribution of velocity fields, evaluation of rugosity and turbulence, and to define the geometrical configuration of the "new" medium.

Then the general laws of Continuum Mechanics are applied to the so defined medium, making proper use of the additional hypotheses. In this way a system of relations between the different variables can be established, which constitute the so called "mathematical model".

Things are now ready for the first calculations with the model, which can be performed by hand or directly in the computer. The first results are used to verify the accuracy of the assumptions which are the basis of the mathematical model.

It goes without saying that the first runs of the model in the computer correspond exactly to the first runs of the physical model. Adjustments and calibration techniques used in physical experiments can also be used to adjust the mathematical model, following a "trial and error" procedure.

We shall not discuss further the possibilities of mathematical models [4]. It is of interest to note that a mathematical model can be deterministic or stochastic. This choice depends, besides physical considerations, on the kind of data available.

We may finally state, from a general point of view, that what we are trying to obtain by means of the mathematical model is a computational scheme which enables us to perform a numerical simulation of the physical problem.

3. — A model for bed-load transportation

If we wish to formulate a mathematical model for the study of scour in open channels and rivers, which may be used to solve scientific and technical steady and unsteady flow problems, it is necessary to discuss first the basic aspects of this phenomenon from an unified point of view.

A general approach to the bed-transportation problem should include:

a) the complete description of the movement of the fluid when viscosity and turbulence are taken into account;

b) the complete description of the movement of the bed of the channel as a continuous medium;

c) the boundary conditions at the contact plane between the fluid and the bed.

The complete description of the fluid movement in open channels may be replaced by an approximate theory based on global equations of Engineering Hydraulics (see for instance reference [5]), which shall be given in this chapter.

According to the principle of conservation of energy, the dynamic equation for unsteady flow reads as follows:

$$\frac{\partial h}{\partial x} - \mathrm{I} + \frac{\alpha \mathrm{V}}{g}\frac{\partial \mathrm{V}}{\partial x} + \frac{1}{g}\frac{\partial \mathrm{V}}{\partial t} + \mathrm{J} = 0 \qquad (2)$$

where $h = h(x, t)$ is the elevation of the water surface with respect to the bottom, V is the mean value of the velocity in the direction of the flow, α is a corrective factor due to the non-uniformity of the velocity field, I is the slope of the bottom of the channel, J is the slope of the energy line E of steady flow and g is the acceleration due to gravity.

The slope J of the energy line can be computed by means of Chezy's formula:

$$\mathrm{J} = \frac{\mathrm{V}^2}{\mathrm{C}^2\mathrm{R}} = \frac{\mathrm{Q}^2}{\mathrm{C}^2\mathrm{F}^2\mathrm{R}} \qquad (3)$$

where R is the hydraulic radius, C is Chezy's coefficient,

$$\mathrm{Q} = \mathrm{VF} \qquad (4)$$

is the total discharge at a generic transverse section of the channel and F is the water area of the same section.

To complete the global equations (2), (3) that govern the movement of the flow, we need the continuity equation:

$$\frac{\partial (\mathrm{VF})}{\partial x} + \frac{\partial \mathrm{F}}{\partial t} = 0 \qquad (5)$$

or alternatively:

$$\frac{\partial \mathrm{Q}}{\partial x} + \frac{\partial \mathrm{F}}{\partial t} = 0 \qquad (5')$$

Note that in the deduction of Eq. (2) it was assumed that both steady and unsteady flow are gradually varied, i.e. the hydrostatic distribution of pressure prevails over the channel section.

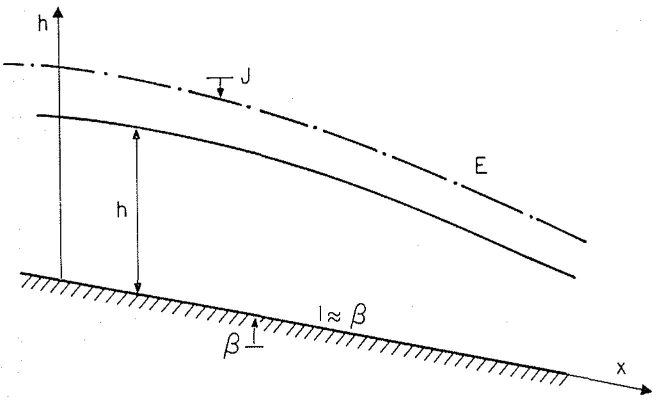
Let us discuss now the simplifications that shall be introduced in the formulation of the scour process of the channel bed. Instead of considering its behaviour as a continuous medium, a global consideration is also used (2). Therefore no boundary conditions at the separation boundary are needed.

(2) This problem has been discussed by Mario H. Gradowczyk : *Una teoría matemática para el estudio de los problemas de erosión*. Publicación N° 10 del Instituto de Cálculo. Buenos Aires (juin 1965).

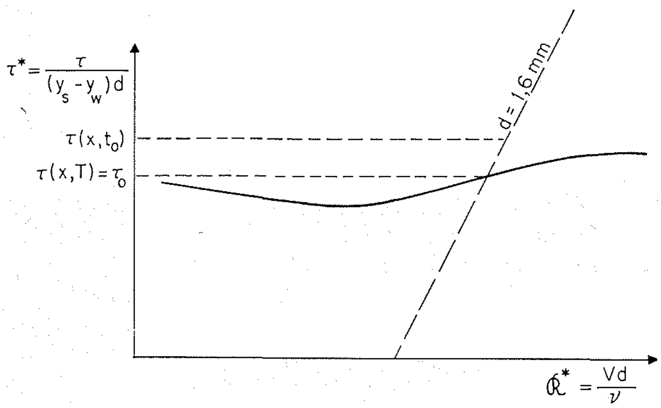
The first global approach to the mechanics of bed-material movement may be traced back to Du Boys [6], who derived a general expression for the total bed-load transport rate per unit width G, which is given in submerged weight. The empirical formulas adopted by many experimentalists, e.g. Meyer-Peter, Schocktlisch have a similar structure as Du Boy's pioneering expression:

$$G = \chi\tau(\tau - \tau_0) \qquad (6)$$

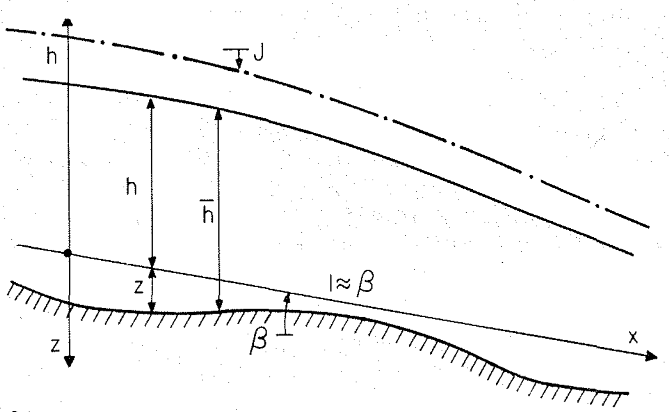
where $\tau = \gamma_w hJ$ is the shear stress transmited by the fluid to the bed material at the contact surface, χ is a coefficient and τ_0 is the critical shear stress below which no entrainment should be expected.



1/



2/



3/

A discussion of the different formulas for bed-load transport usually employed in the literature has been given in references [7], [8].

When the bed material is a homogeneous sand, the Meyer-Peter formula reads as follows:

$$G = 24(\gamma_w RJ - \tau_0)^{3/2} \qquad (7)$$

It is assumed that this formula is also valid for unsteady flow. This hypothesis is similar to other assumptions regarding constitutive equations in Continuum Mechanics, e.g. the validity of Young's modulus for static and dynamic problems.

The critical shear stress τ_0 is an experimental function of the modified Reynolds number

$$\mathcal{R}^* = Vd/\nu,$$

where V is the mean velocity of the flow, d is the diameter of the bed-material particle and ν is the cinematical viscosity of the fluid. τ_0 can be taken from Shields curve, which is shown in Figure 2 and in references [9, 10].

For the general case of unsteady flow and bed-load movement, it is of convenience to rewrite Eq. (2) in a different form:

$$\frac{\partial \bar{h}}{\partial x} - I - \frac{\partial z}{\partial x} + \frac{\alpha V}{g}\frac{\partial V}{\partial x} + \frac{1}{g}\frac{\partial V}{\partial t} + J = 0 \quad (8)$$

where $\bar{h} = \bar{h}(x, t) = h - z$ is the total height of the fluid, $z = z(x, t)$ is the height of the scoured bottom at a time $t = t_0$, which is measured from a system of coordinates which is fixed at a given instant $t = t_0$, as it is shown in Figure 3, and I is the bottom slope at $t = t_0$.

When Eq. (8) is applied to a rectangular streamtube of width b and height h, it can also be given in an alternative form:

$$\frac{\partial \bar{h}}{\partial x} - I - \frac{\partial z}{\partial x} + \frac{\alpha}{2g}\frac{\partial}{\partial x}\left(\frac{Q^2}{b^2h^2}\right) + \frac{1}{g}\frac{\partial}{\partial t}\left(\frac{Q}{h}\right) + J = 0 \quad (8')$$

where Q is the discharge of the streamtube and $J = Q^2/c^2b^2h^3$.

Finally, the continuity equation for the bed-load transport is written down as follows:

$$f(\gamma_s - \gamma_w)\frac{\partial(zb)}{\partial t} + \frac{\partial(Gb)}{\partial x} = 0 \qquad (9)$$

where f is the porosity of the bed material.

We have altogether eight unknown variables V or Q, $\bar{h}$, z, J, G, C, τ_0, τ and five equations (3), (5) or (5′), (7), (8) or (8′), (9). They are supplemented by relations:

$$\tau = \gamma_w RJ \qquad (10)$$

$$\tau_0 = \tau_0(\mathcal{R}^*) \qquad (11)$$

$$C = C_R\left(\frac{R}{d}\right)^{1/6} \qquad (12)$$

where Eq. (11) represents Shields curve and Eq. (12) is the well known Strickler's formula for Chezy's coefficient C. It has been recently shown by Maggiolo and associate that C_R may be considered as a function of the modified Reynolds number $\mathcal{R}^*$ [11]. The use of Strickler's formula, when entrainement is considered, has been recommended by several authors.

It should be stressed that other formulas for G, J and C can be adopted instead of Eqs. (10-12). The final choice of these formulas can be done with the help of physical experiments.

In this way we have established the basic relations governing the unsteady flow in open channels with erodible beds, which we believe is new in this field.

This system of equations may be used to discuss, for example, the influence of erodible beds in the estimation of progressive and roll waves in rivers, predictions of floods, socavation problems around bridge piles. The particular election of V or Q as an independent variable to characterize the motion of the flow, depends upon the problem under study.

To write down the system of relations which constitute the mathematical model, it is necessary to consider an appropriate "new" medium. For this purpose we divide the channel (or portion of the channel under study), into "m" longitudinal streamtubes.

We assume thereof, that Eqs. (8) and (8′) are valid for each streamtube of the channel, hence we take $R = h$ for the interior streamtubes. This is one of the essential features of our mathematical model and can be justified by means of the shallow water theory [3]. This theory shows that Eq. (8) is not restricted to one dimensional problems, i.e. the mean velocity V can be a function of the coordinate perpendicular to the direction of the flow. We believe that this property of Eq. (8) had received little attention in Engineering Hydraulics.

Each streamtube "k" where $1 \leqslant k \leqslant m$, is divided into "$n$" rectangular elements of length $x_i^{(k)}$, height $\bar{h}_i^{(k)}$ and width $b_i^{(k)}$, where $1 \leqslant i \leqslant n$.

No general rule for the choice of the model geometry may be established for the election of the integers "n" and "m". Hence the total number of elementary volumes

$$\bar{h}_i^{(k)} \cdot b_i^{(k)} \cdot \Delta x_i^{(k)}$$

should be established after adequate numerical experimentation. An example of this choice is given in Section 4.1.

To establish the expressions of the model, partial derivatives in the equations (5), (8), (9) are replaced by finite differences. We write down these expressions at an element "i" of the streamtube "k":

$$\bar{h}_{i,j} = \bar{h}_{i-1,j} + z_{i,j-1} - z_{i-1,j-1} - \frac{\alpha}{g} V_{i,j-1} (V_{i,j-1} - V_{i-1,j-1}) - \frac{\Delta x_i}{g \Delta t} (V_{i-1,j} - V_{i-1,j-1}) - \Delta x_i (J_{i-1,j} - I_{i-1}) \quad (13)$$

$$V_{i,j} = \frac{\bar{h}_{i-1,j}\, b_{i-1}}{\bar{h}_{i,j}\, b_i} V_{i-1,j} - \frac{\Delta x_i}{\Delta t}\left(1 - \frac{\bar{h}_{i,j-1}}{\bar{h}_{i,j}}\right) \quad (14)$$

$$C_{i,j} = C_R \left(\frac{\bar{h}_{i,j}}{d}\right)^{1/6} \quad (15)$$

$$J_{i,j} = \frac{V^2_{i,j}}{C^2_{i,j}\, \bar{h}_{i,j}} \quad (16)$$

$$G_{i,j} = 24 (\gamma_w \bar{h}_{i,j} J_{i,j} - \tau_0)^{3/2} \quad (17)$$

$$z_{i,j} = z_{i,j-1} - \frac{\Delta t}{f(\gamma_s - \gamma_w)\, \Delta x_i}\left(G_{i,j} - G_{i-1,j} \frac{b_{i-1}}{b_i}\right) \quad (18)$$

These equations are written in the order as they are calculated. The indices i, j, identify the x and t coordinates respectively. For the sake of simplicity the superscript "k" is dropped in the above expressions.

This computational scheme is adequate for those cases where the unsteadiness of the flow is due principally to the movement of the bottom.

It should be stressed that we are following neither the explicit nor the implicit finite difference method, but we are using the criterium described in section 2, i.e. to reformulate the physical problem considering a simplified mathematical model which can be evaluated numerically in a sequential form. The validity of this approach, which includes both explicit and implicit schemes, can be verified performing numerical experiments as described in section 4.

The mathematical aspects of the numerical integration of the shallow water equations, when the erodible bottom is taken into account, shall be discussed in [14].

4. — Applications

4.1. Establishment of the Uniform Flow in an Open Channel with a Sand Bed.

4.1.1. *General description.*

It shall be discussed here the establishment of the uniform flow in an open prismatic channel of rectangular cross-section, whose flat bottom ($I = 0$) is formed by a sand bed of uniform diameter d.

The physical experiment should be performed in such a way that the flow begins at the initial instant $t = t_0$. This may be achieved in an experimental channel by the movement of a spillway from its initial position to the final one, as it is shown in Figure 4. The discharge per unit width $q = Q/B$ at the spillway may be taken approximately constant with time, where B is the width of the channel.

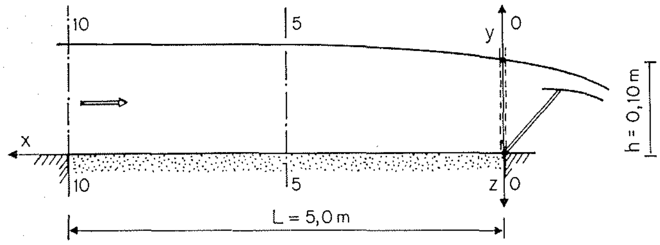
For the establishment of the mathematical model, we shall introduce some simplifications to the general formulation of chapter 3. The channel is considered as a streamtube of unitary width, so that the influence of the walls is not taken into account. It is further assumed that:

$$\frac{\partial \bar{h}}{\partial t} \approx 0$$

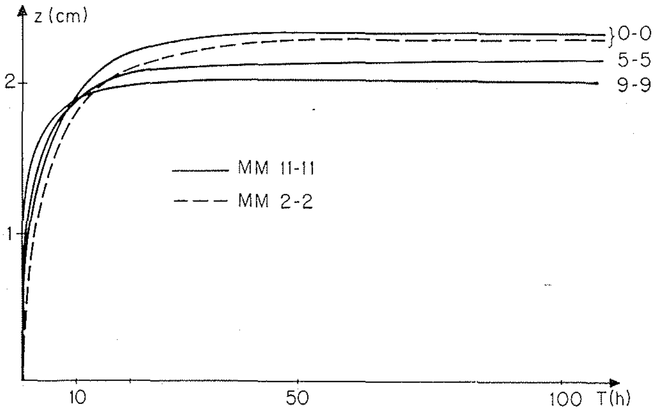
therefore:

$$\frac{\partial q}{\partial x} \approx 0 \text{ i.e. } q(x, t) = q = \text{constant},$$

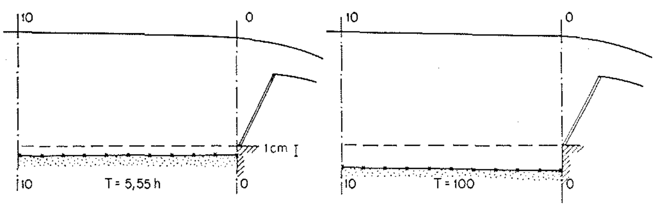
[3] See for instance: J. J. Stoker, *Water Waves*, New York, (1957).



4/



5/



6/

which is also confirmed in physical experiments.

According to these simplifications unsteadiness is due to the erosion of the bed only, and the term:

$$\frac{1}{g}\frac{\partial}{\partial t}\left(\frac{Q}{\bar{h}}\right)$$

in Eq. (8′) may be disregarded.

The equation (13) reduces to:

$$\bar{h}_{i,j} = \bar{h}_{i-1,j} + \frac{1}{[1 - \alpha\,(q^2/g\bar{h}^3{}_{i-1,j})]}(z_{i,j-1} - z_{i-1,j-1} - \Delta x_i \mathrm{J}_{i-1,j}) \qquad (13')$$

It is not necessary to use Eq. (14) because $q = \mathrm{V}\bar{h} =$ constant. The remaining relations of the model do not change.

The corresponding boundary conditions read as follows:

$$h\,(x = 0, t) = \bar{h}\,(x = 0, t) - z\,(x = 0, t) = \text{constant}$$

$$\mathrm{G}\,(x = \mathrm{L}_+, t) = 0$$

$$z\,(x, t = t_0) = 0 \qquad (19)$$

$$\left[1 - \frac{q^2}{g\bar{h}^3\,(x, t = t_0)}\right]\frac{\partial h}{\partial x}\,(x, t = t_0) = \mathrm{J}\,(x, t = t_0)$$

The boundary condition (19.1) results from the constancy of q. The condition (19.2) states that no transport occurs at a section located immediately to the left of section 10-10 (see Fig. 4). The other two correspond to the initial conditions at $t = t_0$.

The election of an appropriate geometry for the mathematical model is of the utmost importance. It is necessary to adopt a geometry which should lead to sufficiently accurate results with a minimum of computing time. As a first approximation, the channel is replaced by a streamtube of length L, so that only two transverse sections 0-0 and 10-10 are considered. This is the so-called two section-model-MM-2.

The physical experiment is performed so that at $t = t_0$ the shear stress at the bottom of the channel $\tau\,(x, t_0)$ is bigger than τ_0, hence the bed of the channel is eroded. As a consequence of the erosion, the mean velocity V decreases and so the shear stress τ. If at $t = \mathrm{T}$, the following inequality is fulfilled: $\tau\,(x, \mathrm{T}) \leqslant \tau_0$, then the bed material reaches an equilibrium position and the entrainment is finished. In this case the flow should become steady and therefore the bottom surface and water surface should be parallel.

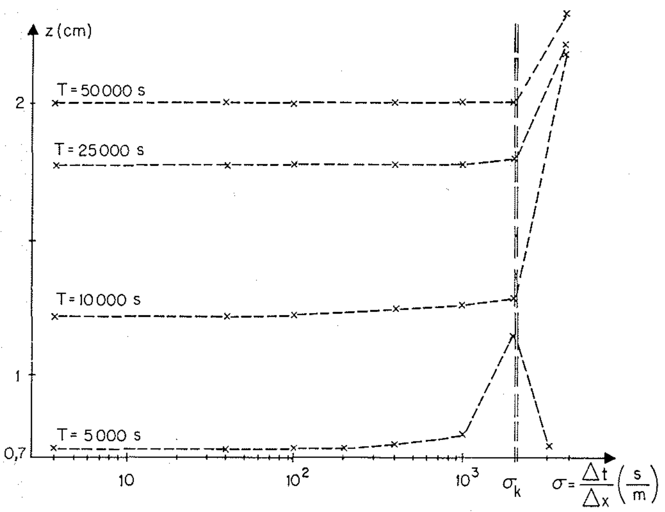
The equilibrium position in a steady flow is calculated by means of the modified Reynolds number and Shields curve, thus leading to the relation $\bar{h}_{\text{final}}/\bar{h}_{\text{initial}} = 1.25$ at section 0-0.

The evolution of the scour phenomenon at section 0-0 of the MM-2 model has been represented with broken lines in Figure 5.

The process begins with a very high erosion velocity which decreases with time. Finally a horizontal asymptote is "practically" obtained at $t = \mathrm{T}$. With the word "practically" we here express that although a horizontal asymptote is impossible to be reached at a finite time T, after a certain finite value $t = \mathrm{T}$ the increase of z is so small that the erosion is measured in quantities ε, ε being much smaller that the grain diameter d of the sand. This has no physical meaning, hence the calculation is stopped at $t = \mathrm{T}$. To check the validity of the MM-2 model, a second mathematical model MM-11 was established, in which the channel was divided in 10 elements of length $\Delta x = \mathrm{L}/10$ (11 sections).

The calculations performed with the MM-11 model are drawn in Figure 5 with full lines showing that the final value of the erosion at section 0-0 ($x = 0$, $t = \mathrm{T}$) calculated by means of the MM-2 is sufficiently accurate. Besides, this new model allows us to follow the erosion process of the bottom of the channel at different times. This cannot be observed from the MM-2 model. The evolution in time of the cross-sections 9-9 and 5-5 is also illustrated in Figure 5.

It is of interest to show here a longitudinal section of the channel at two different times. It can be observed, from the drawings of Figure 6, that when $t = 100$ hours the bottom of the channel is practically parallel to the superficial surface which is almost a recta. Hence the flow may be considered as uniform, which confirms our previous assumptions. It can also be observed from Figure 5 that for $t \geqslant 100$ hs. the curves have practically hori-



7/

zontal asymptotes. The relation $\bar{h}_{\text{initial}}/\bar{h}_{\text{final}}$ results equal to 1.24, which practically agrees with the theoretical prediction given above.

An intermediate model with five elements was also considered, whose results are practically in coincidence with those given by the M-M-11 model, so that they are not represented in Figure 5. This assures the mathematical convergence of the calculations done with the model M-M-11 as well as the validity of the assumptions used in the formulation of model MM-2.

4.1.2. *Convergence.*

The simplifications introduced in 4.1.1. reduce the mathematical formulation of the problem to a system of two partial differential equations.

$$g_1(\bar{h})\frac{\partial \bar{h}}{\partial x} - \frac{\partial z}{\partial x} + J(\bar{h}) = 0$$
$$g_2(\bar{h})\frac{\partial \bar{h}}{\partial x} + \frac{\partial z}{\partial t} = 0 \qquad (20)$$

where $g_1(\bar{h})$, $g_2(\bar{h})$ are known functions of the dependent variable $\bar{h}$. This system of differential equations is of the hyperbolic type.

It is well known, when the method of finite differences is applied to hyperbolic and parabolic partial differential equations, that two different kinds of problems appear: convergence and stability.

The behaviour of the well known hyperbolic equation (wave equation):

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (21)$$

with appropriate boundary conditions, had been studied by many authors. As a result of this analysis, the error of the finite difference method depends, generally, upon the ratio:

$$\frac{\Delta t}{\Delta x} = \sigma$$

where Δt, Δx are the chosen intervals. It should be noted that Eq. (21) is written in non-dimensional quantities so that σ is also non-dimensional.

When the usual explicit computational scheme is adopted and no rounding errors in the computations and in the boundary values are considered, there is a value σ for which an error of the order $(\Delta x)^2$ is achieved. For Eq. (21) this limit value is equal to one.

Although we have formally avoided the numerical integration of Eqs. (20) by the use of the computational scheme which we named the "mathematical model," it is of interest to investigate whether the use of the ratio σ helps in the study of the convergence of our calculations with the model.

We performed a large set of calculations with the MM-11 model, using different values of σ, which are shown in Figure 7. They should allows us to check the influence of σ.

The coordinate z_0 represents the erosion of the section 0-0 ($x = 0$, $t =$ constant). The curves of Figure 7 represent the values of z_0 for different values of σ at different instants $t =$ constant of the process.

It can be observed that the error is larger at the beginning of the entrainment process. The limit value of z_0 when $t \approx$ T is practically independent of the parameter σ, only when σ is smaller than σ_k. This is a critical parameter which determines numerical instability of the computational scheme. This aspect of the problem shall be discussed in section 4.1.3.

No attempt should be done here to show how it is possible to integrate Eqs. (20) by the "method of characteristics". It is clear that this field is a new one so we feel that emphasis should be made first on the fundamental physical ideas rather than on a thoroughly discussion of the mathematical aspects of the problem. This shall be discussed in [14].

4.1.3. *Stability of the mathematical model.*

The numerical instability which appears in the finite-difference solution of hyperbolic and parabolic partial differential equations is due to error propagation, and it appears also in our mathematical model.

The finite-difference schemes can be intrinsically stable, unstable or conditionally stable. This depends on the kind of net chosen.

In the case of the wave equation (21), it can be proved that the explicit difference scheme is stable if $\sigma \leqslant 1$.

The numerical value of the parameter $\sigma = \sigma_k$ which determined the limit of stability in our model was obtained experimentally from our calculations. It is shown in Figure 7 with a broken vertical line.

It could be possible to develop a stability theory for our case, which may determine σ_k in a theoretical form. This objective is however behind the purpose of this paper.

It is of interest to give here a physical explanation of the numerical instability of the mathematical model, which can be useful for practical applications. From the point of view of the calculations performed with the model, as the hydrodynamic condition of the flow is maintained fixed from an instant t to $t + \Delta t$, the corresponding erosion values do not correspond exactly with the real ones. These differences increase for larger values of Δt and may cause, in the next step, a perturbation which changes the physical character of the flow (e.g. negative values of $\bar{h}$).

4.2. Erosion around piles.

The second problem which shall be considered in this paper is the study of the erosion process around an obstacle.

It was the intention of the authors to reproduce by means of the model a large series of experiments on erosion around piles which had been performed by O. J. Maggiolo and associates at the Laboratorio de Mecánica de Flúidos, Facultad de Ingeniería, Montevideo (Uruguay) [12]. We reproduced experiment No. 30, in which a rectangular pile of square cross-section (0.10 × 0.10 mts) is located in the middle of the channel shown in Figure 4.

The determination of the hydrodynamic characteristics of the flow for this particular problem is not an easy task; therefore we established simplificative hypotheses.

It is well known that due to the turbulence of the flow, a surface of separation appears around the pile. Therefore we estimate this surface according to the physical experiments and we then calculate the velocity field and the corresponding streamtubes considering the flow as a plane irrotational one. Due to this simplification the turbulent zone around the pile is not considered in the mathematical model.

As the flow is symmetric, only one half of the channel was considered, which was divided into five streamtubes of twelve elements each, and we applied Eqs (13-18) to each streamtube.

The calculations follow a similar pattern as those of problem 4.1. The unsteadiness of the flow was also neglected.

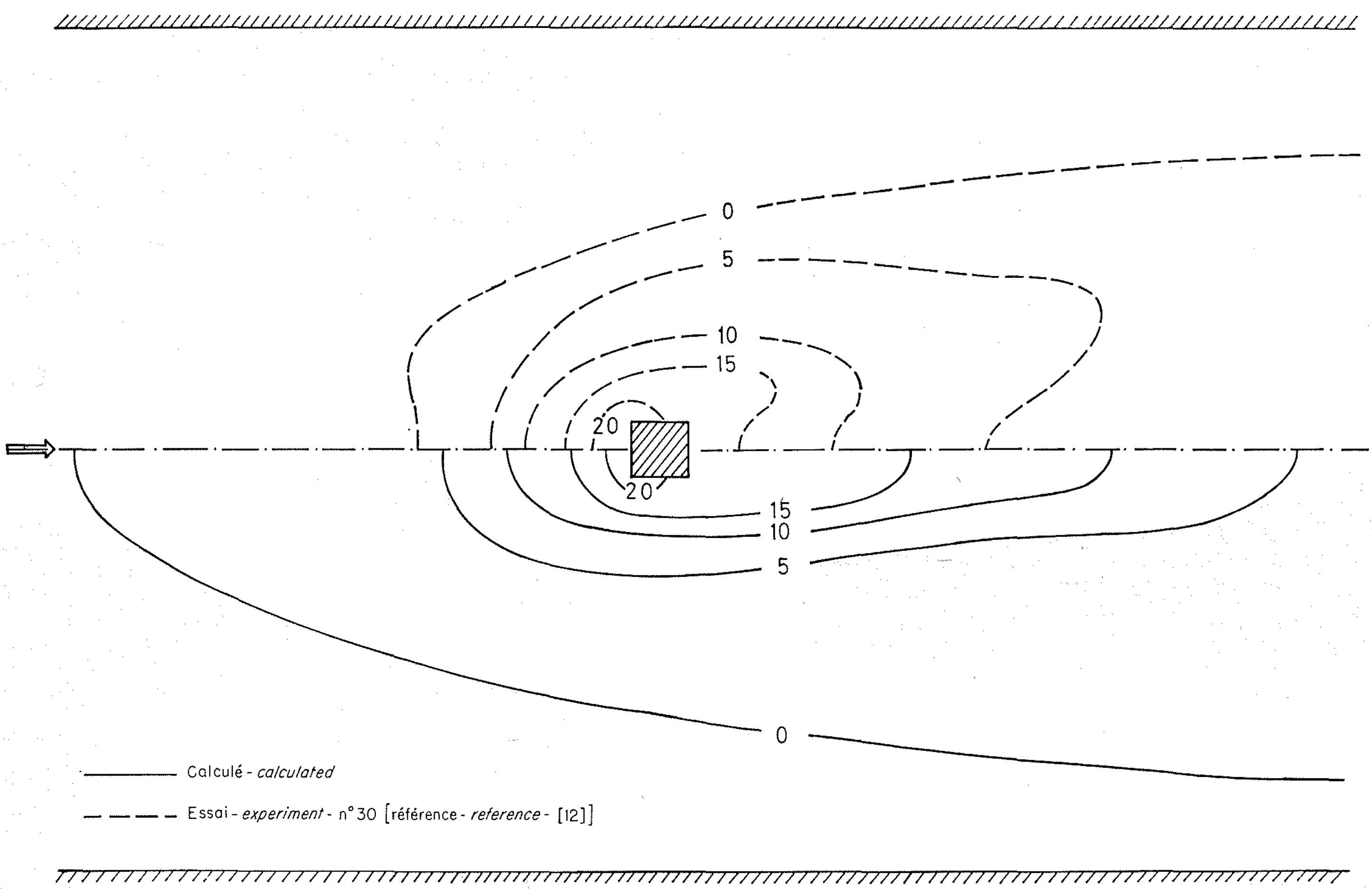
The planimetry of the eroded zone around the pile at $t = 80$ hours, which results from the calculations done with the model, is shown in Figure 8, where the level lines of depth $z =$ constant are given in centimeters.

The experimental results shown in Figure 4 of reference (12) are drawn with broken lines in Figure 8.

It can be observed that good agreement exists between the mathematical model and the physical experience. The evolution of the erosion with time follows also remarkably well the form of the experimental curves.

A good agreement has also been obtained for other experiments of references [12] and [13], even in those cases where the dynamic equilibrium of the socavation process is achieved at a relatively short time.

A complete discussion of these results shall be published elsewhere.



/8/

5. — Mathematical versus physical models

As it was shown in Chapter 4, it is possible to obtain good qualitative as well as quantitative results in a comparatively short time in comparison with physical experiments. So we should like to give here an appreciation of this new technique in comparison with physical models.

ADVANTAGES :

a) Possibility of performing a large series of numerical experiments in a short time, e.g. 80 hours of the physical experiment No. 32 reduces to a few hours of computing time of our model in the Mercury computer. With the faster computers working at present, this time should be reduced to a few minutes or even less.

b) The mathematical model is more flexible to data changes than the physical model, so that a larger number of modifications and changes of geometry, etc. may be performed and processed with practically no delay in time. This is achieved by adequate programming;

c) The results are obtained for any time and for all the field;

d) No distorsions due to the scaling of the model appear. This aspect is of the upmost importance because the last contributions in the field of the bed-load transportation, e.g. reference [10], show that not only the geometric scale should be conserved, but also the modified Reynolds number $\mathcal{R}^*$. This situation is very difficult to reproduce in the physical model;

e) Economic reasons.

DISADVANTAGES :

a) Difficulty in determining the hydrodynamic characteristics of the flow;

b) The obtaining of appropriate expressions for the empirical or experimental coefficients which appear in the model expressions.

The latter disadvantage may be overcome by performing basic physical experiments to determine those physical laws which are difficult to establish in a theoretical form, e.g. the evaluation of the rate of bed-load transportation G.

The former disadvantage may be avoided if the complete equations of the flow are considered. The excellent results obtained by Harlow and Fromm are an example of such an agreement.

It is also possible to use for this purpose special physical models, especially in those cases where obstacles and other singularities appear. These physical models are considerably cheaper than a model for the whole problem.

The convenience of using both techniques together, in those cases in which there is difficulty to obtain the characteristics of the flow, can be concluded from the above considerations. Hence the mathematical model should work as a "simulator" of the total problem, with the advantages of reliability, flexibility and accuracy.

The link of both numerical and experimental techniques constitute and "hybrid system" that should help towards a better understanding of this challenging problem.

It is self-evident that much work need to be done in this direction, and we hope to continue it in an effort to join together engineers, applied mathematicians and physicists in this difficult branch of Mechanics.

ACKNOWLEDGEMENT :

The idea of using mathematical models for erosion processes was suggested by Prof. O. Varsavsky. The authors wish to thank Prof. Maggiolo of the Faculty of Engineering-Montevideo-Uruguay for his interest in this work and specially for giving a Seminar on the physics of the erosion process at the Computing Center of Buenos Aires, in June 1964.

References

[1] GRADOWCZYK (M. H.). — Discusión sobre un modelo matemático para el estudio de los problemas de erosión de lechos móviles. 1° Congreso Latinoamericano de Ingeniería Hidráulica, Porto Alegre (1964); Publicación n° 3 del Instituto de Cálculo, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires.

[2] DIMITRESCU (D.), IONESCU (D. G.), GRACIUM (C.). — L'intégration numérique des équations de Navier-Stokes; Eleventh International Congress of Applied Mechanics, Munich (30 août-5 septembre 1964).

[3] HARLOW (F. H.), FROMM (J. E.). — Dynamics and Heat Transfert in the von Karman Wake of a rectangular cylinder. *The Physics of Fluids*, vol. 7, n° 8 (août 1964).

[4] VARSAVSKY (Q.). — La experimentatión numérica. *Ciencia e Investigación*, t. 19, n° 10 (octobre 1963), Buenos Aires.

[5] CHOW (V.). — Open-Channel Hydraulics. Mc. Graw-Hill, New York, (1959).

[6] DU BOYS (P.). — Le Rhône et les rivières à lit affouillable. *Annales Ponts et Chaussées*, sér. 5 (1879).

[7] ZELLER (J.). — Einführung in den Sedimenttransport offener Gerinne. *Schweizerische Bauzeitung*, 81, Helft 34 (1963).

[8] NING CHIEN. — The present status of research on sediment transport. *Proceedings ASCE*, vol. 80, n° 55 (septembre 1954).

[9] SHIELDS (A.). — Anwendung der Aehnlichkeitsmechanik und der Turbulenzforschung auf die Geschiebebewegung. *Mitteilungen preussische Versuchsanstalt für Wasserbau und Schiffbau* (1936).

[10] CHABERT (J.), CHAUVIN (J. L.). — Formation des dunes et des rides dans les modelés fluviaux. *Bulletin du Centre de Recherches et d'Essais de Chatou*, n° 4 (1963).

[11] MAGGIOLO (O. J.), BORCHI (J.). — Dependencia del coeficiente de Strickler del número de Reynolds y de la rugosidad relativa. Publicación n° 25, Departemento de Mecánica de Fluídos, Facultad de Ingeniería y Agrimensura, Montevideo (Uruguay).

[12] MAGGIOLO (O. J., BORCHI (J.). — Sobre la evolución en el tiempo del proceso de socavación; 1° Congreso Latino-americano de Ingeniería Hidráulica, Porto Alegre (Brasil, 1964).

[13] Etude des affouillements autour des piles de ponts, Laboratoire National d'Hydraulique, Chatou (octobre 1956).

[14] GRADOWCZYK (M. H.), FOLGUERA (H.). — Problemas matemáticos de los regímenes hidráulicos no permanentes sobre fondo móvil (to be published).

Résumé

L'analyse des affouillements dans des canaux découverts à l'aide de modèles mathématiques

par M. H. Gradowczyk * et H. C. Folguera *

Le problème classique des affouillements dans des cours d'eau découverts et l'évolution de leur lit dans le temps sont ici discutés.

Jusqu'à présent on avait réalisé de telles études au moyen, essentiellement, de modèles physiques et de mesures sur place.

La présente communication offre une nouvelle méthode d'aborder ce problème, à l'aide d'une formulation mathématique du processus d'érosion.

Le chapitre 2 est livré à une discussion générale des différentes façons de considérer le problème physique qui trouve son expression dans l'équation (1). Par suite de cette analyse, on introduit la notion d'un modèle mathématique. Ce modèle se traduit par un système de relations qui remplace le problème d'origine (1), et qui peut se calculer plus facilement suivant une forme séquentielle, que selon la méthode classique des différences finies.

On donne au chapitre 3 une description complète de la formulation analytique générale du processus d'érosion. Une telle analyse devrait obligatoirement comporter :

a) Une description complète de l'écoulement dans le cas de la prise en considération de la viscosité et de la turbulence;

b) La description des mouvements du lit du cours d'eau en tant que milieu continu;

c) Les conditions aux limites au plan de contact fluide/lit.

La description de l'écoulement en canal découvert est remplacée par une théorie approchée sur la base des équations globales du génie hydraulique, dont les équations sont : (8) ou (8'); (5) ou (5').

On considère que le fluide ne transporte aucun matériau en suspension.

On utilise une méthode globale également dans le cas du lit du canal. Il n'y a pas besoin de connaître les conditions aux limites à la surface de séparation.

Nous avons retenu la formule de Meyer-Peter (7) pour le taux global de transport solide et nous avons également formulé l'équation de continuité (9) pour le transport solide. Nous avons au total huit paramètres inconnus : V ou Q, h, z, I, G, C, et τ_0 et τ, et cinq équations : (3), (5) ou (5'), (7), (8) ou (8') et (9). Celles-ci sont complétées par les relations (10), (11), (12). Donc, le processus d'érosion est déterminé. Soulignons que d'autres formules sont également valables pour G, I et C.

Ce système d'équations peut servir, par exemple, pour la discussion de l'influence des lits affouillables dans l'évaluation des ondes, des crues et des problèmes d'excavation autour des piles de pont.

Les expressions du schéma de calcul sont explicitées en (13-18), citées dans l'ordre de calcul. Les indices i et j identifient les coordonnées x et t, respectivement. Ce schéma est valable pour les cas d'une instabilité d'écoulement due principalement aux mouvements du lit.

En illustration des applications du modèle, on donne la discusion de deux exemples. En 4.1, on considère l'établissement d'un écoulement uniforme dans un canal découvert ayant un lit de sable (fig. 4). L'évolution dans le temps des phénomènes d'affouillement dans différentes sections du canal est mise en évidence en 5. Les valeurs asymptotes s'accordent bien avec les résultats théoriques. La figure 6 permet d'observer qu'à $t = 100\,h$, le fond du canal est pratiquement parallèle à la surface libre, qui est presque plane. On peut donc considérer que l'écoulement est uniforme.

La convergence, ainsi que la stabilité, du schéma de calcul, sont également traitées.

Au chapitre 4.2, on examine l'érosion autour d'une pile de section rectangulaire. Afin de déterminer les caractéristiques hydrodynamiques de l'écoulement, nous avons évalué la surface de séparation autour de la pile d'après des expériences physiques, puis nous avons calculé le champ des vitesses et les filets d'écoulement correspondants, en employant l'hypothèse d'un écoulement plan et irrotationnel. Ensuite nous avons appliqué les équations (13-18) à chaque filet.

A la figure 8, la planimétrie de la zone affouillée autour de la pile à $t = 80\,h$ est indiquée en traits pleins, où les cotes (z = Cte) sont données en cm. Les résultats expérimentaux (expérience n° 30 de la référence [12]) sont également indiqués, en pointillé.

Enfin, au chapitre 5, on fait le point des avantages et des inconvénients relatifs des deux types de modèles, physique et mathématique. Cette analyse permet de conclure à l'intérêt d'un emploi conjugué des deux techniques. Leur association constitue une « solution hybride », appelée à favoriser une meilleure compréhension de ce problème difficile.

Le modèle mathématique sera donc pris pour la « simulation » du problème dans son ensemble, alors que le modèle physique fournira des renseignements sur des points particuliers.

* Institut de Calcul, Faculté des Sciences exactes et naturelles de Buenos Aires.