LES CLASSIQUES DE LA MÉCANIQUE DES FLUIDES ET DE L'HYDRAULIQUE

SÉRIE PUBLIÉE SOUS LA DIRECTION DE ENZO O. MACAGNO

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Les suggestions concernant les textes à inclure dans cette série seront les bienvenues, spécialement si des indications précises sont données, dans le cas d'articles très longs ou de livres, sur les parties considérées comme les plus importantes.

CLASSICAL WORKS IN FLUID MECHANICS AND HYDRAULICS

A SERIES SELECTED BY ENZO O. MACAGNO

No attempt to correct errors, if they would be detected, will be made, unless they appear as obvious misprints. Each reader will be confronted with the original writing as it was. Translations in this series are intended to be quite literal with the purpose of providing an access as direct as possible to the original form of the writing.
Suggestions to include material in this series will be most welcome, especially if indications are given of the excerpts that are considered valuable in the case of long papers or books.

LEONHARD EULER

(1707-1783)

GENERAL PRINCIPLES OF THE STATE OF EQUILIBRIUM OF FLUIDS

Mémoires de l'Académie des sciences de Berlin, 11 (1755), 1757, p. 217-273

Translation of p. 217-226

GENERAL PRINCIPLES OF FLUID MOTION

Mémoires de l'Académie des sciences de Berlin, 11 (1755), 1757, p. 274-315

Translation of p. 274-287

FURTHER RESEARCH ON THE THEORY OF FLUID MOTION

Mémoires de l'Académie des sciences de Berlin, 11 (1755), 1757, p. 316-361

Translation of p. 316-325 and p. 341-345

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This translation was cheked by Dr. Emmet O'Loughlin, Research Engineer at the Institute of Hydraulic Research of the University of Iowa.

GENERAL PRINCIPLES OF THE STATE OF EQUILIBRIUM OF FLUIDS

COMMENTATIO 225 INDICIS ENESTROEMIANI
MÉMOIRES DE L'ACADÉMIE DES SCIENCES DE BERLIN 11 (1755), 1757, p. 217—273
LEONHARDI EULERI OPERA OMNIA, SER. SECUNDA, VOL. XII, P. 2-53, 94 ARTICLES *.

1. My aim here is to expound the principles on which the whole of Hydrostatics, or the Science of the equilibrium of the fluids, is founded. To give these principles their greatest possible extent, I shall include in my research not only fluids having the same degree of density everywhere—such as water and other liquid bodies—of which it is said that they do not suffer any compression, but also those fluids which are composed of particles of a different density, whether this difference suits them by virtue of their own nature, or whether it results from forces pressing the particles against one another. Air should quite clearly be included in this latter kind, as well as those other fluid bodies that are called elastic. Moreover, I shall not limit my research to the force of gravity alone, but I shall extend it to any force such that each particle of the fluid may be acted upon.

2. Such is the plan that I have set out to accomplish, which clearly shows first of all that the common principles of Hydrostatics whose explanation is found in the elements are no more than a very special case of those I shall establish here. For, on the one hand, one commonly considers only gravity, to the action of which the particles of the fluid are subjected, while, on the other hand, one considers only the fluids of the first kind, all the parts of which retain the same degree of density everywhere. And though one has not neglected to study thoroughly the state of equilibrium of elastic fluids—and in particular that of air—the principles which have been established therefrom seem to be so different from the former that one is barely able to reduce them to a common origin based on the nature of fluids in general.

3. Although I am proposing such a wide generalization with respect both to the nature of the fluid and to the forces acting on every one of its particles, I do not fear the kind of reproach that has often—and justifiably—been levelled at those who have taken it upon themselves to carry the research of others to a more general level. I do appreciate that over-generalization often obscures rather than throws fresh light on a subject, and that it sometimes leads to calculations so involved as to make it exceedingly difficult to deduces therefrom consequences for even the simplest cases. Where generalisations are burdened with this disadvantage, it is quite certainly very much better to desist from making them and to confine one's research to special cases.

4. However, exactly the contrary happens in the subject I am proposing to expound: rather than dazzle our intellect, the generality I am embracing will expose for us the true laws of Nature in all their splendour, whilst revealing still more powerful reasons for admiring their beauty and simplicity. It will be an important lesson to learn that principles one thought were associated with some particular case are in fact much more extensive in their application. Moreover, this research will demand calculations hardly more cumbersome than before, and it will be easy to apply them to any special case one may consider.

* Attention is directed to the excellent paper by C. A. TRUESDELL, "Rational Fluid Mechanics, 1687-1765", included in vol. XII of *L. Euleri Opera Omnia, ser. secunda*. See specially Truesdell's comments on papers Nos. 225, 226, 227.

5. In fact, this is all a matter of clearly establishing the first idea on which all the reasoning we shall need to achieve our aim will have to be founded: the idea of the nature of fluidity in general. For the laws governing the equilibrium of fluids can only differ from those applying to solid bodies in so far as the natures of fluids and solids differ. The question, therefore, is to identify the true and essential difference which distinguishes fluids from solids, but despite the controversy between Philosophers and Physicists on the question, we cannot deduce from what they have said about it anything of any use for our purpose. Though it may be true that there is no connection between the smallest particles in a fluid, and that they are in a continuous state of motion, this truth would be absolutely sterile with respect to the research in question, which requires a much deeper study of the nature of fluid bodies.

6. Inasmuch as this essential property of fluids must yield the principles of Hydrostatics, I am only able to detect it in the quality whereby we know that a fluid mass cannot be in a state of equilibrium unless all parts of its surface be acted upon by forces all equal and perpendicular to that surface. I am supposing here that the particles inside the fluid mass are not acted upon by any forces, for if indeed they were, the external forces would have to balance them, and they would also all have to be equal to one another. Thus, I am considering a fluid mass that is not subjected to any force, and about which there is no doubt that it is in equilibrium. Now, let forces be imagined acting externally on its surface; in order that the mass be maintained in equilibrium, the forces must be perpendicular to it and all equal to one another, and they must act on all the elements of its surface. Moreover, if the fluid be elastic, its elasticity must be equal to the forces acting upon its mass, in order that the latter neither extends into a greater volume nor is reduced to a smaller one.

7. This property makes the most essential distinction between fluid and solid bodies. A solid body can be held in equilibrium when acted upon by two equal and contrary forces, and it does

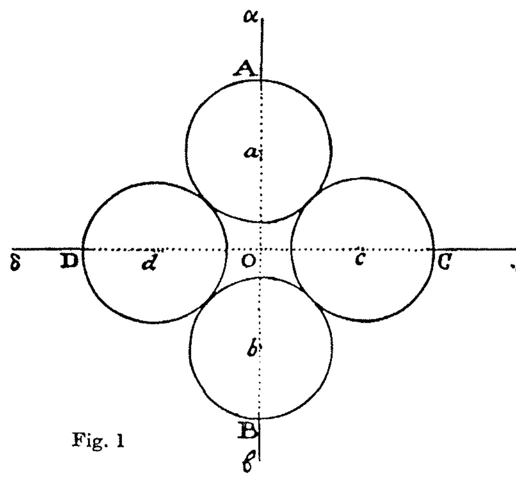

Fig. 1

not exert on the adjoining parts a tendency to escape. Now, a mass (Fig. 1) of several solid bodies that are not joined to one another is already a closer approach to the nature of a fluid, as is shown by the case of four contiguous spheres *a*, *b*, *c*, *d*, for although the two opposite spheres *a* and *b* be pressed by equal and opposite forces *A* and *B*, no state of equilibrium will exist; the two other spheres, however, will be pushed by them in directions *C* and *D*, by forces which are in the ratio of the distance *cd* to the distance *ab*. Hence, in order that these four spheres remain in equilibrium, one must add forces *C* and *D* to forces *A* and *B*. If there were several spheres or other solid bodies, preservation of equilibrium would require several further forces, depending on their number and respective positions.

8. If these spheres be assumed infinitely small and their number infinitely great, it may then happen that the state of equilibrium requires that an infinite number of forces act upon this mass from all sides, so that if any one such force were missing, the state of equilibrium would be destroyed. One could also imagine an arrangement among these corpuscles such as to result in all the forces required for equilibrium becoming equal to one another, which would represent exactly the case of a fluid. However, not only would such a case be, so to speak, morally impossible, but as soon as the least change occurred, the forces needed for equilibrium would not fail to become exceedingly unequal to one another; instead, the equality of these forces is preserved always and necessarily, in spite of any changes undergone by the fluid. It follows clearly from this that fluidity cannot be explained as a mass consisting of an infinitely large number of loose particles, even if these are assumed to be infinitely small, and it also seems highly doubtful whether any internal motion could make up for this insufficiency.

9. This, therefore, is what constitutes the nature of fluidity, i.e. that a fluid cannot be in equilibrium unless it is pressed upon from all sides by forces that are all equal and perpendicular to its surface. Thus (Fig. 2), when a fluid mass *ABCDEF* is pressed upon at a place *AB* by any force *PM*, whose direction is perpendicular to a portion of the surface *AB* upon which it acts, and if one also conceives some other portion *CD*, then in order that the fluid be maintained in equilibrium, that portion *CD* must also be pressed upon perpendicularly by a force *QN* standing in the same ratio to the former force *PM* as the ratio between the areas *CD* and *AB*. If one of these forces were less than according to this ratio, it would not be sufficient to withstand the action of the other, and the state of equilibrium would therefore be upset. The same is true for any other portion of the fluid's surface—disregarding gravity and any other force which might act immediately upon the particles of the fluid.

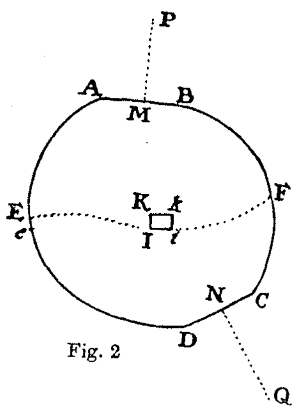

Fig. 2

10. It follows from this that if one knows the pressure acting upon a plane on the surface of the fluid, one also knows the pressures on all the parts of the surface that are required for equilibrium. Thus, if we write for the base $AB = aa$, and P for the force pressing against it, any other base $CD = cc$ will be pressed against by the force $= \frac{cc}{aa} P$. This rule becomes simpler if we express the force P by the weight of a cylinder of some homogeneous heavy material whose base is $= aa$, that is to say the base on which this force acts. This cylinder will have a certain height, which we shall call p; consequently, the force P will be equal to the weight of a mass of the said homogeneous material, the volume of which is aap, or one can write $P = aap$: hence, since the force which must act upon the base $CD = cc$ is $= \frac{cc}{aa} P$, it will become $= ccp$ and will thus be equal to the weight of a cylinder of the same homogeneous material whose base is $= cc$ and whose height is the same as before, p. For the same reason, any other portion of the surface ff of this fluid mass will sustain a force $= ffp$.

11. Hence, to know the state of the pressures whereby a fluid mass is maintained in equilibrium, it suffices to know this height p which is common to all cylinders formed of this homogeneous heavy material by means of whose weights we are expressing here the forces acting upon the fluid. This is so because if one knows this height p, the force pressing on each portion of the fluid surface is easily determined: thus, considering a portion aa, this force will be expressed by the weight aap. As this force acts perpendicularly to the surface at all points, one evidently cannot draw any immediate conclusion as to the force being sustained by a convex or a concave portion of the surface; it will therefore be necessary to resort to infinitely small surface elements; if such an element be $= ds^2$, the force pressing upon it will be $= pds^2$ and its direction will perpendicular to the element, wich can always be looked upon as being flat.

12. To understand better the force of this pressure, let it be imagined that the fluid is enclosed in a vessel having at *AB* an opening $= aa$ filled by a piston being acted upon perpendicularly by a force $PM = aap$; this being so, the fluid will press equally everywhere against the vessel walls, so that if there were an opening *Ee* somewhere, the fluid would escape through it. Now, if the hole *Ee* the area of which is, for instance, $= ee$ be stopped up to prevent the fluid from escaping, a force $= eep$ must be applied perpendicularly to it, from which one then knows the forces which each element of the inside surface of the vessel sustains from the pressure $PM = aap$ acting on the base $AB = aa$. If this base were less and the pressing force were also less by the same proportion, the pressure would nevertheless remain the same on the vessel, whence it is seen that the smallest force *PM* is capable of producing as great a pressure within the vessel as one wishes, provided the base $AB = aa$ be made sufficiently small so that the height p in the expression of force aap becomes as great as one wishes.

13. However, with a fluid in such a state of pressure under the action of some force $PM = aap$, not only do all the vessels elements sustain pressures corresponding to the same height p, but all the elements of the fluid itself are also in the same state of pressure. Let an immaterial diaphragm *ELiF* be assumed to lie within the fluid, separating some portion *AEFB* from the remainder of the fluid mass; and, since this portion is in equilibrium, all the particles of the diaphragm will also be sustaining forces complying with the same height p. From this, it follows that each fluid element *IKki* will be equally compressed in all directions by such forces; in other words, all the fluid particles will be compressed by forces due to the same height p; thus, it is the equality of all these forces which constitutes the state of equilibrium, always assuming that there are no particular forces—such as gravity—acting on the fluid particles.

14. By this reasoning, one is able to form a correct idea of what I call the state of pressure of a fluid; and this pressure could not be more adequately represented than by a certain height related to the weight of a homogeneous material, which one will judge as being the most appropriate to be used for purposes of such measurement. Thus, when I say that the state of pressure of the fluid element *IKki* is expressed by the height p, it should be understood that each face of this element, which I shall take as $= ds^2$, is pressed upon by a force that is equal to the weight of a cylinder of the said homogeneous material, the base of which is $= ds^2$ and the height of which is $= p$. This height p thus expresses the force with which the neighboring fluid elements act from all sides upon the element *IKki*, and therefore the force with which the element reacts on those surrounding it. Thus it is also by this same force that the element *IKki* resists the compression which would reduce it to a smaller volume, so that if its resistance were less, there would actually be a reduction in volume.

15. This consideration brings us to the distinction between elastic and non-elastic fluids, or compressible and non-compressible fluids, though the state of equilibrium we have just explained applies equally to both. For if the fluid enclosed in vessel *ABCDEF* be elastic or compressible, the force $P = aap$ acting on the piston *AB* would reduce the fluid to such a degree of compression in which it would be in equilibrium; clearly, therefore, the elasticity of the fluid is precisely equal to the compressing force, i.e., the height p will also serve as a measure for the elasticity of the fluid. If the elasticity were greater than the height p, the piston would be pushed back to a state of equilibrium; if it were smaller, the piston would penetrate deeper: as the fluid can neither extend to infinity nor reduce itself into a vanishing space, there will always be a case in which equilibrium must occur.

16. From this, one understands that when an compressible fluid is reduced into a lesser volume, its elasticity must become greater as the more one wishes to compress the fluid, the greater the force one must employ. Elasticity thus necessarily depends on the density of the fluid in the sense that the more the density increases, the greater also does elasticity become; of course, it is not necessary that the elasticity be proportional to the density. One also notices this in air, namely that elasticity is not exactly proportional to density. However, when the changes are not considerable and far removed from both the greatest volume and the smallest to which the fluid can be reduced, it can be assumed that elasticity is exactly proportional to density. Now, it can happen that another quality in addition to density also plays a part in determining elasticity, such as heat for example, which increases the springiness of air, though its degree of density remains the same. But if the heat differs, one can embody the effect of this in the proportion existing between elasticity and density, which then becomes a variable by virtue of this fact.

17. Hence, if a fluid mass is in equilibrium and the pressure in it is expressed by the height p, this same height will also be a measure of elasticity, and by the ratio existing between density and elasticity one will also know the density, and vice versa. Thus, if the density of the fluid is $= q$, and Q marks the function thereof to which the elasticity would be proportional if heat—or any other quantity influencing the spring—were invariable, that fluid could not be in equilibrium unless the pressure p be as Q. Now let it be assumed that heat, or some other variable quantity, be $= r$, with r expressing the spring under a given density, the pressure required for equilibrium would be as Qr, or more generally as a certain function of q and r. Let this function be Π, the value of which becomes $= \Gamma$ in a determined case, in which $q = g$ and $r = h$; hence, if elasticity is expressed in this case by the height f, the proportion

$$f : p = \Gamma : \Pi$$

will give pressure or elasticity for any other case

$$p = \frac{f\Pi}{\Gamma};$$

an expression which, by its general nature, applies to all imaginable cases.

18. It may happen that a very small change in density can produce a very big change in elasticity, so that the fluid does not undergo any appreciable change in volume if the pressure p is very considerably increased or reduced, and that it conserves nearly the same density, and when this small change vanishes entirely, we will have precisely the case of a non-compressible fluid which, without changing either its volume or its density, is able to sustain any pressure, however great it may be. In this case, therefore, the function Π must vanish, as well as its determinate value Γ, in order that the fraction $\frac{f\Pi}{\Gamma}$ become indeterminate. In other words, the density q, which can generally be considered to be a function of the elasticity p will in this case become a constant quantity. One understands from this how very inappropriate it is to describe these fluids as non-elastic, for on the contrary, they contain all possible degrees of elasticity under the same density, whereas the fluids one calls elastic also contain all possible degrees, but each under a different density.

19. The sole idea of pressure I have just established and represented by a height contains all that belongs to the knowledge of fluid equilibrium. For one knows first of all the forces which the fluid exerts on the vessel containing it, and if it so happens that any part of those forces vanish, equilibrium will prevail without the fluid being confined in that space. In the second place, if a solid body be immersed in a fluid, one can determine the forces pressing against that body from all sides, and thence the forces the body sustains from the fluid. In the third place, if the parts of the fluid are subject to compression and the ratio between density and elasticity are known, as this is everywhere equal to pressure, on will be able to establish the fluid density at any point. And all these questions one may raise on the equilibrium of fluids can easily be reduced to these three articles, which will supply the solutions thereof.

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GENERAL PRINCIPLES OF FLUID MOTION

COMMENTATIO 226 INDICIS ENESTROEMIANI
MÉMOIRES DE L'ACADÉMIE DES SCIENCES DE BERLIN 11 (1755), 1757, p. 274—315
LEONHARDI EULERI OPERA OMNIA, SER. SECUNDA, VOL. XII, P. 54-91, 68 ARTICLES.

1. Having established in my previous "Mémoire" the principles of fluid equilibrium more generally both as regards the varied quality of fluids and the forces that are able to act in them, I now propose to deal with fluid motion on the same footing and attempt to establish the general principles on which the entire science of fluid motion is founded. It will be readily understood that this is a much more difficult matter and that it involves incomparably more profound research; nevertheless, this too I hope to be fortunate enough to master, so that if any difficulties remain they refer not to the mechanical but solely to the analytical aspect for this science has not yet been brought to the degree of perfection that would be necessary to expand the analytical formulae containing the principle of fluid motion.

2. The aim, therefore, must be to discover the principles whereby one can determine the motion of a fluid irrespective of both its state and the forces acting upon it. For this, let us examine in detail all the articles forming the subject of our research and containing both known and unknown quantities. And first of all, the nature of the fluid whose various types are to be considered is assumed to be known: thus, it is either incompressible or compressible. If it is incompressible, one must distinguish between two cases, one in which the entire mass consists of homogeneous parts whose density is and always remains the same everywhere, and the other in which the mass is composed of heterogeneous parts; in this case, one must know both the density of each species and the proportions of the mixture. If the fluid is compressible and its density variable, one must know the law according to which its elasticity depends on density, also whether its elasticity depends only on density or on some other quality as well, such as heat, which is peculiar to each particle of the fluid at least for every instant of time.

3. One must also assume that the state of the fluid at a certain time—which I shall call the "primitive state of the fluid" —is known; as this state is to all intents and purposes an arbitrary one, the first things to establish are how the particles of which the fluid consists are arranged and the motion impressed upon them, unless the fluid was at rest when in its primitive state. But the primitive motion is not entirely arbitrary: both the continuity and impenetrability of the fluid impose a certain limitation which I shall attempt to establish later on. However, one often knows nothing of a primitive state, as for example where the motion of a river is to be determined, and the object then becomes merely to find the permanent state the fluid will ultimately reach without undergoing further changes. Now it so happens that neither this circumstance nor the primitive state change anything in the research one will have to do and the calculation will always remain the same: it is only in integrations where this must be taken into account in order to determine the constants accompanying each integration.

4. In the third place, one must include among the data the external forces acting upon the fluid: I am calling these forces "external" to distinguish them from the internal forces with which the fluid particles act upon each other, as these are the main subject of research subsequently to be carrier out. One may assume, therefore, that the fluid is not acted upon by any external forces, or only by natural gravity, which is considered to act with the same magnitude and direction everywhere. Now, to make this research more general, I shall consider the fluid as being subjected to arbitrary forces which may either be directed towards one or several centres or obey some other law as regard both their quantity and direction. Only the accelerating actions of these forces are known immediately, without consideration of the masses on which they act. I shall therefore only introduce the accelerating forces into the calculation, from which it will be a simple matter to deduce the true driving forces, by multiplying the accelerating forces by the masses acted upon in each case.

5. Let us now consider the articles containing that which is unknown. Now, in order to really know the motion by which the fluid will be carried, one must determine for every instant and place both the motion and the pressure of the fluid; moreover, if the fluid is compressible, one must also define its density whilst knowing that other quantity which, with density, defines elasticity; which being counter-balanced by the pressure of the fluid, must be assumed to be equal to it, as in the case of equilibrium, in which connection I have expounded these ideas with greater care. Thus, the number of quantities involved in determining this motion is seen to be much greater than in the case of equilibrium, as it is necessary to introduce symbols denoting the motion of each particle, and because all these quantities may vary in time. Thus, in addition to the symbols determining the position of every conceivable point in the fluid mass, it is also necessary to introduce one to account for the time elapsed, and which, by its variability, can be applied to any proposed time.

6. Hence, let t be the time elapsed after a primitive state (Fig. 1), and let the fluid now be in a state of motion, which is to be determined. Irrespective of the space occupied by the fluid at the present time, I begin by considering an arbitrary point Z within the fluid mass; then, to introduce the position of this point Z into the calculation, I refer it to three fixed axes, OA, OB and OC, which are perpendicular to each other at point O, and whose positions are given. Let OA and OB lie in the plane of the sketch, and let OC be perpendicular to that plane. Also, let the perpendicular ZY be drawn from point Z to the plane AOB, and the normal YX from point Y to the axes OA, in order to obtain cordinates $OX = x$, $XY = y$ and $YZ = z$ parallel to our three axes. These coordinates x, y and

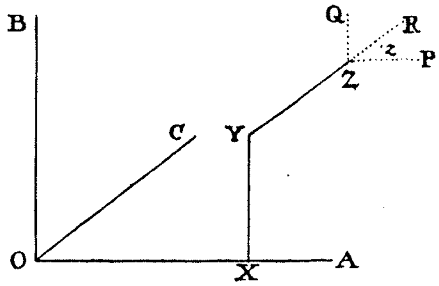

Fig. 1

z will have definite values for each point considered within the fluid mass; by giving these three coordinates all possible positive and negative values in succession, one will cover all the points in the infinite space, and hence also those within the space occupied by the fluid at every instant of time.

7. In second place, I consider the accelerating forces acting during the present instant upon the fluid particle at Z; now, irrespective of what these forces may be, they always can be reduced to three acting in directions ZP, ZQ and ZR parallel to our three axes OA, OB and OC. Then, denoting the accelerating force of natural gravity as unity, let P, Q and R be the accelerating forces acting on point Z in directions ZP, ZQ and ZR: these letters P, Q and R will denote absolute numbers. If the forces acting at the same point of space Z are always the same, quantities P, Q and R will be expressed by certain functions of coordinates x, y and z; however, if these forces also varied with the time t, they would also include the time t. Now, I assume that these functions are known, as one must include the active forces among the known quantities, whether they depend solely on the variables x, y and z, or as well on the time t.

8. Now, let r denote the heat at point Z, or that other quality which, with density, governs elasticity in the case of a compressible fluid; r must also be considered as a function of variables x, y, z and of time t, as it might change in time at the same point of the space Z; this function can thus be regarded as being known. Then, let the present density of the fluid particle at $Z = q$, taking unity for the density of a certain homogeneous material which I shall use to measure pressure with the aid of heights, as explained at greater length in my "Mémoire" on fluid equilibrium [1]). Also, for the present let the pressure of the fluid at point Z be expressed by the height p, which will thus also denote the elasticity; and since the nature of the fluid is assumed to be known, one will also know the relationship between height p and quantities q and r. Now, p and q will also be functions of the four variables, x, y, z and t, but will be unknown; but when the fluid is not compressible [2]), the pressure p is independant of the density q, and the other quality r does not come into consideration at all.

9. Finally, whatever may be the motion of the fluid element at Z at the present time, it can be resolved into the directions ZP, ZQ and ZR parallel to our three axes. Thus, let u, v and w be the velocities of this motion resolved into the three directions ZP, ZQ and ZR; these three quantities clearly must also be considered as functions of the four variables x, y, z and t. For if, having established the nature of these functions, if one then takes time t as being constant, the variability of coordinates x, y and z will give the three velocities u, v and w, and hence the true motion of each fluid element at the present instant; and if coordinates x, y and z are considered to be constant, and time t to be the only variable, one will find the motion, not of a certain fluid element, but of all the elements passing through the same point Z in succession, or one will know the motion of that fluid element which will then be at point Z, at any instant of time.

10. However, let us also see which path the fluid element follows which is now at Z for an infinitely short time interval dt, or at which point it will be an instant later. Now, if we express space as the product of velocity and time, the fluid element now at Z will be carried in direction ZP through the space udt, in direction ZQ through the space vdt, and in direction ZR through the space wdt. If, therefore, we write:

$$ZP = udt; \quad ZQ = vdt \quad \text{and} \quad ZR = wdt$$

and one completes the parallelepipedon on these three sides, the angle opposite Z will mark the point at which the fluid element in question will be after time dt, and the diagonal of this parallelepipedon, which is

$$= dt\sqrt{(uu + vv + ww)}$$

will give the true path followed, from which the velocity of this true movements will be

$$= \sqrt{(uu + vv + ww)}$$

and the direction will then easily be determined by the sides of this parallelepipedon, for it will be inclined with respect to the plane AOB at an angle whose sine is

$$= \frac{w}{\sqrt{(uu + vv + ww)}},$$

with respect to the plane AOC at an angle whose sine is

$$= \frac{v}{\sqrt{(uu + vv + ww)}},$$

and, finally, with respect to the plane BOC at an angle whose sine is

$$= \frac{u}{\sqrt{(uu + vv + ww)}},$$

11. Having determined the motion of the fluid at Z at the present instant, let us now also examine that of some other infinitely close element, and let this element be at point z whose coordinates are the following

$$x + dx, \quad y + dy \quad \text{and} \quad z + dz.$$

The three velocities of this element in the directions of the three axes will thus be given by the quantities u, v, w, after substitution of $x + dx$, $y + dy$ and $z + dz$ for each respectively, or after addition of their differentials by assuming time t to be constant. If $x + dx$ be substituted for x, the increments of u, v and w are:

$$dx\left(\frac{du}{dx}\right), \quad dx\left(\frac{dv}{dx}\right) \text{ and } dx\left(\frac{dw}{dx}\right);$$

and if $y + dy$ be substituted for y, the increments are

$$dy\left(\frac{du}{dy}\right), \quad dy\left(\frac{dv}{dy}\right) \text{ and } dy\left(\frac{dw}{dy}\right);$$

and the same also holds good for the variability of z. Hence, the three velocities of the fluid element, which is now at z, will be the following:

along OA

$$= u + dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right) + dz\left(\frac{du}{dz}\right)$$

along OB

$$= v + dx\left(\frac{dv}{dx}\right) + dy\left(\frac{dv}{dy}\right) + dz\left(\frac{dv}{dz}\right)$$

along OC

$$= w + dx\left(\frac{dw}{dx}\right) + dy\left(\frac{dw}{dy}\right) + dz\left(\frac{dw}{dz}\right)$$

12. These are the velocities that are appropriate for a fluid element at z infinitely close to point Z, the locus of which is determined by the three coordinates:

$$x + dx, \quad y + dy \quad \text{and} \quad z + dz.$$

1) See previous "Mémoire". C.T.
2) "incompressible" in the original version published. C.T.

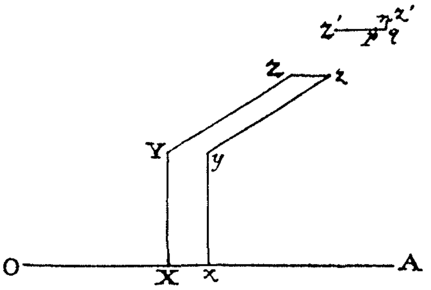

Fig. 2

Thus, if we take point z (Fig. 2) in such a manner that only x changes by dx and the other two coordinates y and z remain the same as for point Z, the three velocities of the fluid element at this point z will be:

$$u + dx\left(\frac{du}{dx}\right),\ v + dx\left(\frac{dv}{dx}\right),\ w + dx\left(\frac{dw}{dx}\right)$$

by which this element will be carried in the time dt to another point z' whose locus with respect to point Z' we must define, which would be that to which the fluid element that was at Z is carried within the same space of time dt and whose locus was determined above (paragraph 10). In order to identify this point z', I observe that if the velocities of z were exactly the same as those of Z, point z' would lie at p, so that the distance Z p would be equal and parallel to the distance Zz. And since, by the hypothesis, Zz is parallel to axis OA and equal to dx, line $Z'p$ will also $= dx$ and be parallel to axis OA.

13. Now, as the velocity along OA is not u, but $u + dx\left(\frac{du}{dx}\right)$, due to this difference the element in question will be carried from p to q in the direction $Z'p$, so that:

$$pq = dtdx\left(\frac{du}{dx}\right),$$

it would therefore be at q if the other two velocities were v and w. But as the velocity along axis OB is:

$$v + dx\left(\frac{dv}{dx}\right),$$

this difference will carry our element from q to r, through the space:

$$qr = dtdx\left(\frac{dv}{dx}\right),$$

and parallel to axis OB. And finally, the increment $dx\left(\frac{dw}{dx}\right)$ of velocity w will transport the element from r to z', by the particle:

$$rz' = dtdx\left(\frac{dw}{dx}\right),$$

and parallel to the third axis, OC, whence, I conclude that the fluid element which occupied the small straight line Zz will be transported in time dt to line $Z'z'$, which will lie at an infinitely small angle to axis OA, and whose length, because:

$$Z'q = dx\left(1 + dt\left(\frac{du}{dx}\right)\right)$$

will be:

$$dx\sqrt{\left(\left(1 + dt\left(\frac{du}{dx}\right)\right)^2 + dt^2\left(\frac{dv}{dx}\right)^2 + dt^2\left(\frac{dw}{dx}\right)^2\right)}.$$

Therefore, neglecting terms containing the square of dt, the length of $Z'z'$ will not be different from $Z'q$, so that:

$$Z'z' = dx\left(1 + dt\left(\frac{du}{dx}\right)\right);$$

All that need be said about the inclination of this line with respect to the axis OA is that it is infinitely small of the first degree, or expressed as αdt.

14. If the small line Zz had been taken as $= dy$ and parallel to axis OB, one would find by the same reasoning that the fluid which occupied this line had been transported to another one.

$$Z'z' = dy\left(1 + dt\left(\frac{dv}{dy}\right)\right)$$

whose inclination with respect to axis OB would also be infinitely small. And if one took the line $Zz = dz$, and parallel to the third axis, OC, the fluid occupying it would be transported to another line:

$$Z'z' = dz\left(1 + dt\left(\frac{dw}{dz}\right)\right)$$

whose inclination with respect to axis OC would be infinitely small (Fig. 3). If, therefore, we consider a rectangular parallelepipedon $ZPQRzpqr$ formed by its three sides:

$$ZP = dx,\quad ZQ = dy \quad \text{and} \quad ZR = dz,$$

the fluid which occupied this space will be transported in time dt to fill the space Z' P' Q' R' z' p' q' r', differing infinitely little from a rectangular parallelepipedon whose three sides will be:

$$Z'P' = dx\left(1 + dt\left(\frac{du}{dx}\right)\right),$$

$$Z'Q' = dy\left(1 + dt\left(\frac{dv}{dy}\right)\right),$$

$$Z'R' = dz\left(1 + dt\left(\frac{dw}{dz}\right)\right).$$

Because the sides ZP, ZQ, ZR are transported to Z' P', Z' Q, Z' R', there is no doubt at all that the fluid content of the first space is also transported to the other space in time dt.

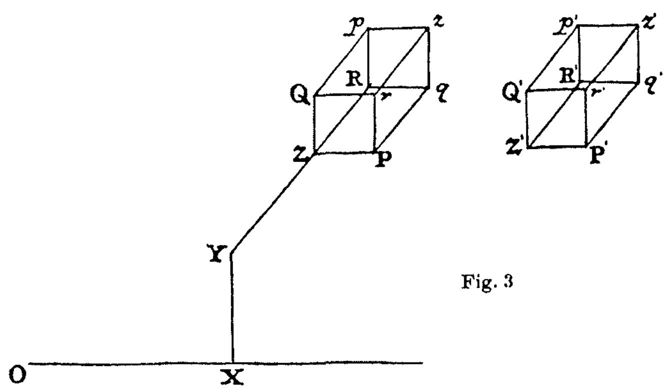

Fig. 3

15. One can now see whether the volume of fluid which occupied parallelepipedon Zz has become greater or smaller after time dt: one merely needs to find the volume or the capacity of each of these two solids. Now, as the former solid is a rectangular parallelepipedon formed by sides dx, dy, dz, its volume is $= dx\,dy\,dz$; for the other one, however, whose plane angles differ infinitely little from a right angle, I observe that its volume can also be found by multiplying these three sides; for the error resulting from the infinitely small obliquity will be contained in terms in which the time element dt would rise

to two dimensions, so that it is permissible to neglect it. This volume $Z'z'$ will therefore be expressed as follows:

$$dx\,dy\,dz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)$$

Anyone still with any lingering doubts about the reliability of this conclusion should read my "Pièce latine" "Principia motus fluidorum", in which I have calculated this volume without neglecting anything [1]).

16. Therefore, if the fluid is not compressible, these two volumes must be equal to one another, as the mass which occupied space Zz could occupy neither a larger space nor a smaller one. But as I propose to approach this matter in as general a manner as possible, and as I have named the density at $Z = q$, considering q to be a function of the three coordinates and of time, I observe that, in order to find the density at Z', one must first increase time t by its differential dt, and then, as the locus Z' is different from Z, one must increase quantities x, y, z by the small spaces udt, vdt, wdt; hence, the density at Z' will be:

$$q+dt\left(\frac{dq}{dt}\right)+udt\left(\frac{dq}{dx}\right)+vdt\left(\frac{dq}{dy}\right)+wdt\left(\frac{dq}{dz}\right),$$

and hence, as densitiy is inversely proportional to volume, this quantity will be to q as $dxdydz$ is to:

$$dx\,dy\,dz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right).$$

Consequently dividing by dt, we shall obtain this equation, which consideration of density yields:

$$\left(\frac{dq}{dt}\right)+u\left(\frac{dq}{dx}\right)+v\left(\frac{dq}{dy}\right)+w\left(\frac{dq}{dz}\right)$$
$$+q\left(\frac{du}{dx}\right)+q\left(\frac{dv}{dy}\right)+q\left(\frac{dw}{dz}\right)=0.$$

17. Here, then, is a truly remarkable condition, which already establishes a certain relationship between the three velocities u, v, and w with respect to the density of the fluid q. Now, this equation can be reduced to one of greater simplicity: for $u\left(\frac{dq}{dx}\right)$ is not different from $\left(\frac{udq}{dx}\right)$, as this method of expression is meant to convey that, in the differentiation of q, x is the only quantity considered as a variable; hence, similarly:

$$q\left(\frac{du}{dx}\right)=\left(\frac{qdu}{dx}\right);$$

from which it is evident that:

$$q\left(\frac{du}{dx}\right)+u\left(\frac{dq}{dx}\right)=\left(\frac{udq+qdu}{dx}\right)=\left(\frac{q.du}{dx}\right),$$

taking the differential of the product qu such that one considers the quantity x as the only variable. This is why the equation we have found reduces to the following:

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

If the fluid were not compressible, the density would be the same at Z and Z', and in this case, one would have this equation:

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0,$$

which is also the one on which I based Latin "Mémoire" referred to above.

18. Having been obtained by consideration of the continuity of the fluid, this formula already contains a certain relationship which must exist between the quantities u, v, w and q. The other determinations must be deduced from the consideration of the forces acting on each fluid particle: in addition to the accelerating forces P, Q, R acting on the fluid at Z, it also undergoes the pressure acting from all sides on the element of the fluid contained in Z. Three accelerating forces in the directions of the three axes are obtained from the combination of these double forces; and as the actual accelerations can be assigned by considering the velocities u, v, and w, we shall thus obtain three equations which, together with that we have just found, will include everything concerning the motion of fluids, so that we shall then be in possession of the general and complete principles for the entire science of fluid motion.

19. All we need to find the accelerations to which the fluid element at Z is subjected is to compare the velocities u, v and w at the present time at point Z with those after time interval dt at point Z'. A double change thus occurs, concerning the coordinates x, y, z, which receive increments udt, vdt, wdt, on the one hand, and time, which increases by dt, on the other hand. It follows that the three velocities meeting at point Z' are:

in direction OA

$$=u+dt\left(\frac{du}{dt}\right)+udt\left(\frac{du}{dx}\right)+vdt\left(\frac{du}{dy}\right)+wdt\left(\frac{du}{dz}\right),$$

in direction OB

$$=v+dt\left(\frac{dv}{dt}\right)+udt\left(\frac{dv}{dx}\right)+vdt\left(\frac{dv}{dy}\right)+wdt\left(\frac{dv}{dz}\right),$$

in direction OC

$$=w+dt\left(\frac{dw}{dt}\right)+udt\left(\frac{dw}{dx}\right)+vdt\left(\frac{dw}{dy}\right)+wdt\left(\frac{dw}{dz}\right).$$

And hence, the accelerations expressed as the velocity increments divided by the time element dt will be:

in direction OA

$$=\left(\frac{du}{dt}\right)+u\left(\frac{du}{dx}\right)+v\left(\frac{du}{dy}\right)+w\left(\frac{du}{dz}\right),$$

in direction OB

$$=\left(\frac{dv}{dt}\right)+u\left(\frac{dv}{dx}\right)+v\left(\frac{dv}{dy}\right)+w\left(\frac{dv}{dz}\right),$$

in direction OC

$$=\left(\frac{dw}{dt}\right)+u\left(\frac{dw}{dx}\right)+v\left(\frac{dw}{dy}\right)+w\left(\frac{dw}{dz}\right).$$

20. Now, let us try to find the accelerating forces in these same directions, resulting from the pressures exerted by the fluid on parallelepipedon Zz, whose volume is $= dx\,dy\,dz$, and hence the mass of the fluid occupying it $= q\,dx\,dy\,dz$. Now, the pressure at point Z being expressed by the height p, the driving force face $ZQRp$ receives from it is $= p\,dy\,dz$; and, for the opposite face, $zqrP = dy\,dz$, the height p is increased by its differential $dx\left(\frac{dp}{dx}\right)$, which results from the assumption that x is the only variable. Thus, this fluid mass Zz is pushed back in direction AO by the driving force:

$$dx\,dy\,dz\left(\frac{dp}{dx}\right),$$

1) See "Mémoire" 258 in this Volume, read before the Berlin Academy on the 31st August, 1752. C.T.

or by the accelerating force $= \frac{1}{q}\left(\frac{dp}{dx}\right)$. Similarly, it will be seen that the fluid mass Zz is acted upon in direction BO by the accelerating force $= \frac{1}{q}\left(\frac{dp}{dy}\right)$, and in direction CO by the accelerating force $= \frac{1}{q}\left(\frac{dp}{dz}\right)$. Adding to these forces the given quantities P, Q, R, the whole accelerating forces will be:

$$\text{in direction } OA = P - \frac{1}{q}\left(\frac{dp}{dx}\right),$$

$$\text{in direction } OB = Q - \frac{1}{q}\left(\frac{dp}{dy}\right),$$

$$\text{in direction } OC = R - \frac{1}{q}\left(\frac{dp}{dz}\right).$$

21. All we need to do, therefore, is to equate these accelerating forces with the actual accelerations we have just found, and we shall obtain the three following equations:

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right),$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right),$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right).$$

If, first, we add to these three equations the one we found by consideration of the continuity of the fluid:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0,$$

and then the one giving the relationship between elasticity p, density q, and that other quality r which affects elasticity p, in addition to density q, we shall have five equations which contain the entire theory of fluid motion.

22. Whatever the nature of forces P, Q, R, provided they are real, it should be noted that:

$$Pdx + Qdy + Rdz$$

is always a real differential of a certain finite determinate quantity, assuming the three coordinates x, y and z to be variable; consequently, the following will always hold good:

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right), \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right), \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right)$$

and if we put this finite quantity $= S$, so that:

$$dS = Pdx + Qdy + Rdz$$

assuming time t to be constant, should forces P, Q, R also vary with time at the same places; this quantity S gives what I call the effort of the acting forces, and is the sum of the whole value of each force multiplied by the element in its direction, or by the small distance over which a body would be dragged by the action of the force. This idea of the effort is of the utmost importance in the theory of both equilibrium and motion, because it was been shown that the sum of all the efforts is always either a *maximum* or a *minimum*. This beautiful property goes admirably well with the fine principle of the least action, for the discovery of which we are indebted to our Illustrious President, Monsieur de Maupertuis.

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FURTHER RESEARCH ON THE THEORY OF FLUID MOTION

COMMENTATIO 227 INDICIS ENESTROEMIANI

MÉMOIRES DE L'ACADÉMIE DES SCIENCES DE BERLIN 11 (1755), 1757, p. 316—361

LEONHARDI EULERI OPERA OMNIA, SER. SECUNDA, VOL. XII, P. 92-132, 80 ARTICLES.

1. Having reduced the entire Theory of fluid equilibrium and motion to two analytical equations in my two previous "Mémoires", the consideration of these formulae seems to be of paramount importance as they not only contain all that has already been discovered on the subject of fluid equilibrium and motion by methods which differ widely and most of which are not particularly convincing, but also anything one may still desire in this particular Science. Sublime though the fluid research may be for which we are indebted to Messrs. Bernoulli, Clairaut and d'Alembert, it follows so naturally from my two general formulae that one could not easily stop admiring this agreement between their profound meditations and the simplicity of the principles from which I derived my two equations, and to which I was led immediately by the first axioms of Mechanics.

2. Although it is seldom advisable to extend our research too far, lest one become involved in a calculation so complicated as to be only very laboriously applicable to the simplest cases, exactly the contrary happens here: although my equations are general, they are nevertheless simple enough to be applied to any particular case: by virtue of this very fact, they make such universal truths available to us that our knowledge benefits from the greatest enlightenment one might wish for. And at the same time, despite the greater universality they embrace, they are still almost as simple as when one considers particular cases.

3. I have already remarked that my formulae contain the entire Theory of both equilibrium and motion of fluids; moreover, with reference to the nature of fluids, they also apply as much to fluids known as elastic fluids as to those which cannot undergo compression; and regarding the former, irrespective of the manner in which density may depend on elasticity, be it according to some constant law, or to one that is variable in one way or another. Further, whatever the accelerating forces acting on the fluid elements, their effect is also included in the said formulae, and finally, whatever the external forces acting on the fluid, and whatever the configuration of the canal or vessel containing the fluid, they allow for all these different circumstances.

4. If may be that the question is one of equilibrium, or of the motion of a fluid, and that either the velocity and direction of each particle be called for, or the forces exerted by the fluid upon the sides of the vessel containing it, or the resistance assayed by a solid body immersed in it, or the elasticity and density of the fluid if the latter be compressible everywhere: all these questions and others similar that one can imagine on the subjects of equilibrium and motion of fluids, reduce to a single research, which is that into the state of pressure in which the fluid is at each point. I measure this pressure by the height of a column of a homogeneous heavy material whose density I put $= 1$, so that in order to find the pressure sustained by an infinitely small surface, one merely needs to multiply that surface by the appropriate height for it, and the weight of that volume, being filled with the said homogeneous material, will be equal to the pressure sought.

5. It is for this pressure, or rather the height whereby it is measured, that I give a differential equation, and everything comes down to finding its integral. But as this equation contains several variables, one is unable to integrate it until one has found the relationship between these variables which is necessary in order that the equation may become integrable. This yields the conditions for the entire motion, both with respect to the velocity of each particle and to the density within it at each point and at each instant; consequently, a single differential equation contains several different determinations all at the same time. Integration by itself only yields the determination of pressure, but integrability takes the place of several equations, which supply the other essential determinations for the Theory of fluid motion. Now, in order to obtain all the determinations whereby the motion is determined entirely in each case, it is necessary to adjoin to this equation the other equation I have found.

6. This other equation can be regarded as finite, as it does not contain differentials, though it does include relations thereof. It is founded on the continuity of the fluid, and excludes the void which the fluid particles might leave between them, and also their mutual penetration. This latter circumstance is as essential to fluid bodies as to solid ones; however, with regard to the other, it may well happen that the fluid particles actually separate, leaving a void between them, as we can see in jets of water, which eventually dissipate into drops. As the parts are then entirely separate from each other, it stands to reason that one would be unable to apply my formulae to them; unless one were willing to consider each drop separately, as constituting a separate fluid body. The entire Theory of fluids is thus founded on but two equations, one of which contains pressure, and the other the continuity of the fluid, in all its parts.

. . . .

15. Now, for abbreviation I write:

$$X = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right),$$

$$Y = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right),$$

$$Z = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right).$$

and the differential equation which determines pressure p is:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - Xdx - Ydy - Zdz,$$

in which time t is assumed to be constant. Now, the other equation obtained from the continuity of the fluid is:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0,$$

and it is these two equations which contain the entire Theory of equilibrium and motion of fluids, in the greatest universality one could possibly imagine.

. . . .

18. Others who have approached this matter, if one excepts Mr. d'Alembert, have only given the fluid an extension by no more than two dimensions, or they have at least assumed the motion of each particle to take place in the same plane, so that one cannot regard the formulae they have found as anything but particular; instead those I have just given are quite general, and one could not imagine any case, however complicated, which would not be included in them. It will be a good thing to show first of all, therefore, that all which has been yet discovered about the motion of fluids can very easily be deduced from my general formulae; now, nearly all that has been given on this subject comes down to the motion of fluids through infinitely narrow pipes, or at least which can be regarded as such, so that one only conceives a single dimension in both the fluid and its motion in such cases. Then, I shall also show how everything that has been written about the motion of fluids while considering two dimensions, follows quite naturally from these same formulae.

. . . .

46. The motion of the fluid is assumed to be steady; all the particles passing successively through point Z will describe the same path. ...... Let us therefore introduce this line *FZV* into the calculation ......

. . . .

52. This same integral $\left(\frac{p}{g} = V - \frac{1}{2}\,88 + D\right)$ * is also found if we do not neglect the formulae:

$$\left(\frac{du}{dy}\right), \left(\frac{du}{dz}\right) \text{ etc.}$$

provided we observe that, for the fluid velocities within the line *FZV*, there are the following:

$$udy = vdx, \quad udz = wdz \quad \text{and} \quad vdz = wdy$$

For then the formula $Xdx + Ydy + Zdz$ changes into this one:

$$\left.\begin{array}{l} udx\left(\frac{du}{dx}\right) + udy\left(\frac{du}{dy}\right) + udz\left(\frac{du}{dz}\right) \\ vdx\left(\frac{dv}{dx}\right) + vdy\left(\frac{dv}{dy}\right) + vdz\left(\frac{dv}{dz}\right) \\ wdx\left(\frac{dw}{dx}\right) + wdy\left(\frac{dw}{dy}\right) + wdz\left(\frac{dw}{dz}\right) \end{array}\right\} =$$

$$= 8\,dx\left(\frac{d8}{dx}\right) + 8\,dy\left(\frac{d8}{dy}\right) + 8\,dz\left(\frac{d8}{dz}\right)$$

the integral of which is clearly $\frac{1}{2}\,88$, and we shall obtain the following, as before:

$$\frac{p}{g} = V - \frac{1}{2}\,88 + D.$$

. . . .

53. Hence, still considering the same curved line *FZV*, as the value of D is constant we shall be able to compare with each other the fluid pressures at all points of the line, so that if we know the pressure at a single point, we can deduce the pressure at all the others from it ......

* $V = S$; see previous "Mémoire" (No. 22). 8 denotes the true fluid velocity at Z.