COMPUTATIONAL STABILITY OF EXPLICIT DIFFERENCE FORM OF EQUATIONS FOR VISCOUS FLUID FLOW

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During a computational investigation of viscous-fluid flow in conduit expansions, a preliminary study has been made of the stability and accuracy of the calculations. This analysis was based on applying the same numerical techniques to a flow with separation and to a purposely disturbed uniform flow. It was conjectured that the behaviour of the discretized form of the equations would not be too different for uniform and nonuniform conduits, and it could then be established with a great economy of computer time by studying the simpler case. The conjecture seemed very plausible, because experimental knowledge indicated that steady flow with separation at not-too-low Reynolds numbers consisted of a weak eddy—or two—and a main flow with a smooth transition from one uniform region to another. The results obtained to date for expansions of ratio 2:1 confirm the supposition.

To effect the disturbance of the uniform flow, a length of conduit slightly greater than one conduit width was subdivided by means of a square-mesh net; the location of a mesh point was indicated by the pair of integers i, j. The values of the stream function and the vorticity were then calculated for all the i, j points according to the know exact solutions for two-dimensional and axisymmetric Poiseuille flows. The values of the vorticity were numerically disturbed by a factor $(1 + e)$ at inner points with an even value of i, and by $(1 - e)$ at odd values of i; the values of e used were 0.1, 0.2, 0.50, 0.75, and 1.00. The result of this operation was considered as a vorticity distribution hypothetically in error at the start of a computation.

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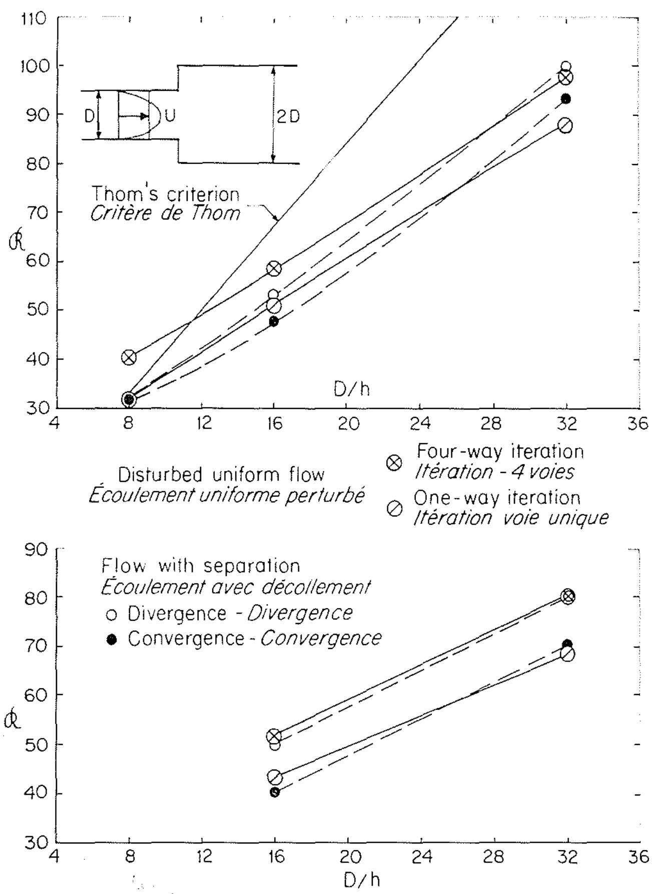
The finite-difference formulas utilized were of the same type for all conduit geometries considered; only the formulas for the circular tube are given here:

$$\overset{K+1}{\eta}_{i,j} = [1 - \mathcal{R}\,(\overset{K}{\psi}_{i+1,j} - \overset{K}{\psi}_{i-1,j})/2\,h_0 + 4\,r_{i,j}^2/h_0^2]^{-1}$$
$$\times \left\{ \begin{array}{l} r_{i,j}^2\,(\overset{K}{\eta}_{i+1,j} + \overset{K+1}{\eta}_{i-1,j} + \overset{K+1}{\eta}_{i,j-1} + \overset{K}{\eta}_{i,j+1})/h_0^2 \\ + r_{i,j}\,(\overset{K+1}{\eta}_{i,j-1} - \overset{K}{\eta}_{i,j+1})/(2\,h_0) \\ - \mathcal{R} r_{i,j}\,[(\overset{K}{\psi}_{i,j+1} - \overset{K}{\psi}_{i,j-1})\,(\overset{K}{\eta}_{i+1,j} - \overset{K+1}{\eta}_{i-1,j}) \\ + (\overset{K}{\psi}_{i+1,j} - \overset{K}{\psi}_{i-1,j})\,(\overset{K+1}{\eta}_{i,j-1} - \overset{K}{\eta}_{i,j+1})]/(4\,h_0^2) \end{array} \right\}$$

$$\overset{K+1}{\psi}_{i,j} = \frac{1}{4} \left\{ \begin{array}{l} \overset{K}{\psi}_{i+1,j} + \overset{K+1}{\psi}_{i-1,j} + \overset{K+1}{\psi}_{i,j-1} + \overset{K}{\psi}_{i,j+1} \\ - h_0^2\,[r_{i,j}\,\overset{K+1}{\eta}_{i,j} + (\overset{K+1}{\psi}_{i,j-1} - \overset{K}{\psi}_{i,j+1})/(2\,r_{i,j}\,h_0)] \end{array} \right\}$$

$$\eta_B = \frac{\psi_B - \psi_{B+1} + \dfrac{h_0^2}{6}\,r_{B+2}\,\eta_{B+1} + \dfrac{h_0^3}{12}\,\eta_{B+2}}{\dfrac{h_0^2}{3}\left(\dfrac{h_0}{4} + \dfrac{h_0^2}{8\,r_B} - r_B\right)}$$

herein, ψ is the streamfunction; η, the vorticity; h_0, the mesh size; r, the radial distance; $\mathcal{R}$, the Reynolds number. All quantities are dimensionless. B indicates points on the wall. K is the iteration index. The first formula is a discretized form of the vorticity diffusion equation, while the second expresses the vorticity, and the third the nonslip condition at fixed walls. Of the many different ways in which the field can be swept when iterating the previous formulas, two were tried. In the first, the field was always swept, say, from the SW corner of the field to the NE corner; in the second, from



1-2/ Two-dimensional flow and axisymmetric flow. Points below full lines are numerically stable.
Ecoulement plan et écoulement axisymétrique. Les points situés au-dessous des lignes continues sont numériquement stables.

the SW corner to the NE corner and back, then from the SE corner to the NW corner and back, and so on until the results either settled within an assigned margin or diverged.

As should be expected, the accuracy improved with increased number of meshes. Constant residual errors, however, were found to persist after the numerical solution was settled through a large number of iterations. Those residual errors were small enough to consider the result satisfactory; all other things being equal, larger residual errors resulted for the axisymmetric flow than for the two-dimensional flow. Findings concerning the stability of the numerical solution are represented in Figures 1 and 2. Results of calculations for conduit expansions are also shown; they consume an order of magnitude more time, and therefore only few of these computations have been performed. The line separating stable from unstable points is thus not so sharply defined as in the case of disturbed uniform flow, but it must certainly be within the band defined by the dashed lines in both figures.

From this study it has been found that the influence of the method of sweeping the field is not negligible; the four-way iteration (with practically the same programming and computing time) is definitely more advantageous than the one-way iteration. Within the range of disturbances investigated a tenfold increment of the assumed initial deviations did not affect the results to a noticeable degree. The most important result is that the analysis of the purposely disturbed uniform flow gives a reliable indication of the mesh size that ensures computational stability for a given Reynolds number in a flow with a region of separation. It remains to be determined how much more a nonuniform conduit can depart from the uniform before the indication is invalidated.