LES CLASSIQUES DE LA MÉCANIQUE DES FLUIDES ET DE L'HYDRAULIQUE

SÉRIE PUBLIÉE SOUS LA DIRECTION DE ENZO O. MACAGNO

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Les suggestions concernant les textes à inclure dans cette série seront les bienvenues, spécialement si des indications précises sont données, dans le cas d'articles très longs ou de livres, sur les parties considérées comme les plus importantes.

CLASSICAL WORKS IN FLUID MECHANICS AND HYDRAULICS

A SERIES SELECTED BY ENZO O. MACAGNO

No attempt to correct errors, if they would be detected, will be made, unless they appear as obvious misprints. Each reader will be confronted with the original writing as it was. Translations in this series are intended to be quite literal with the purpose of providing an access as direct as possible to the original form of the writing.
Suggestions to include material in this series will be most welcome, especially if indications are given of the excerpts that are considered valuable in the case of long papers or books.

GEORGE GABRIEL STOKES

(1819-1903)

ON THE THEORIES OF THE INTERNAL FRICTION OF FLUIDS IN MOTION, AND OF THE EQUILIBRIUM AND MOTION OF ELASTIC SOLIDS

Read, April 14, 1845

Transactions of the Cambridge Philosophical Society, Vol. 8, 1845

Excerpts from the introduction and section I

In his "Report on recent researches in hydrodynamics", presented in 1846 to the British Association, Stokes included comments on the derivations of the equations of motion of viscous fluids due to Navier, Poisson, Saint-Venant, and himself. Those comments are appended to these excerpts of his classic paper on the subject.

Dans le « Rapport sur les recherches récentes en hydrodynamique », qu'il présenta en 1846 à la "British Association", Stokes commentait les travaux de Navier, Poisson, Saint-Venant, ainsi que les siens propres, sur l'établissement des équations du mouvement des fluides visqueux. Ces commentaires sont publiés ci-après, à la suite des extraits de son mémoire classique sur le sujet.

XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids. By* G. G. Stokes, M.A., *Fellow of Pembroke College.*

[Read *April* 14, 1845.]

The equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former *. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another; or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

. . . .

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of equations (13), and of the principles on which they are formed. I afterwards found that Poisson had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society +. The leading principles of my theory will be found in the hypotheses of Art. 1, and in Art. 3.

Section I.

Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases.

Art. 1. — . . . .

. . . .

Let P be any material point in the fluid, and consider the instantaneous motion of a very small element E of the fluid about P. This motion is compounded of a motion of translation, the same as that of P, and of the motion of the several points of E relatively to P. If we conceive a velocity equal and opposite to that of P impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about P; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of P.

. . . .

. . . . I shall assume the following principle: —

That the difference between the pressure on a plane in a given direction passing through any point P *of a fluid in motion and the pressure which would exist in all directions about* P *if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about* P; *and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.*

. . . .

Art. 2. — . . . .

. . . .

Let u, v, w be the resolved parts, parallel to the rectangular axes, Ox, Oy, Oz, of the velocity of the point P, whose co-ordinates at the instant considered are x, y, z. Then the relative velocities at the point P', whose co-ordinates are $x+x'$, $y+y'$, $z+z'$, will be

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z' \text{ parallel to } x,$$
$$\frac{dv}{dx}x' + \frac{dv}{dy}y' + \frac{dv}{dz}z' \;\ldots\ldots\ldots\; y,$$
$$\frac{dw}{dx}x' + \frac{dw}{dy}y' + \frac{dw}{dz}z' \;\ldots\ldots\ldots\; z,$$

neglecting squares and products of x', y', z'. Let these velocities be compounded of those due to the angular velocities ω', ω'', ω''' about the axes of x, y, z, and of the velocities U, V, W parallel to x, y, z. The linear velocities due to the angular velocities being $\omega''z' - \omega'''y'$, $\omega'''x' - \omega'z'$, $\omega'y' - \omega''x'$ parallel to the axes of x, y, z, we shall therefore have

$$U = \frac{du}{dx}x' + \left(\frac{du}{dy} + \omega'''\right)y' + \left(\frac{du}{dz} - \omega''\right)z',$$
$$V = \left(\frac{dv}{dx} - \omega'''\right)x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \omega'\right)z',$$
$$W = \left(\frac{dw}{dx} + \omega''\right)x' + \left(\frac{dw}{dy} - \omega'\right)y' + \frac{dw}{dz}z'.$$

Since ω', ω'', ω''' are arbitrary, let them be so assumed that

$$\frac{dU}{dy'} = \frac{dV}{dx'}, \quad \frac{dV}{dz'} = \frac{dW}{dy'}, \quad \frac{dW}{dx'} = \frac{dU}{dz'},$$

which gives

$$\omega' = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \; \omega'' = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \; \omega''' = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right), \ldots\ldots$$

* This may be easily shewn by the consideration of a tetrahedron of the fluid, as in Art. 4.

+ The same equations have also been obtained by Navier in the case of an incompressible fluid (*Mém. de l'Académie*, t. VI. p. 389), but his principles differ from mine still more than do Poisson's.

$$\left.\begin{aligned}U &= \frac{du}{dx}x' + \tfrac{1}{2}\left(\frac{du}{dy}+\frac{dv}{dx}\right)y' + \tfrac{1}{2}\left(\frac{du}{dz}+\frac{dw}{dx}\right)z',\\ V &= \tfrac{1}{2}\left(\frac{dv}{dx}+\frac{du}{dy}\right)x' + \frac{dv}{dy}y' + \tfrac{1}{2}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)z',\\ W &= \tfrac{1}{2}\left(\frac{dw}{dx}+\frac{du}{dz}\right)x' + \tfrac{1}{2}\left(\frac{dw}{dy}+\frac{dv}{dz}\right)y' + \frac{dw}{dz}z',\end{aligned}\right\}\ldots(2).$$

The quantities ω', ω'', ω''' are what I shall call the *angular velocities of the fluid* at the point considered...............

....

Let us now investigate whether it is possible to determine x', y', z' so that, considering only the relative velocities U, V, W, the line joining the points P, P' shall have no angular motion. The conditions to be satisfied, in order that this may be the case, are evidently that the increments of the relative co-ordinates x', y', z', of the second point shall be ultimately proportional to those co-ordinates. If e be the rate of extension of the line joining the two points considered, we shall therefore have

$$\left.\begin{aligned}Fx' + hy' + gz' &= ex',\\ hx' + Gy' + fz' &= ey',\\ gx' + fy' + Hz' &= ez',\end{aligned}\right\}\ldots\ldots\ldots\ldots(3);$$

where

$$F=\frac{du}{dx},\quad G=\frac{dv}{dy},\quad H=\frac{dw}{dz},\quad 2f=\frac{dv}{dz}+\frac{dw}{dy},$$

$$2g=\frac{dw}{dx}+\frac{du}{dz},\quad 2h=\frac{du}{dy}+\frac{dv}{dx}.$$

If we eliminate from equations (3) the two ratios which exist between the three quantities x', y', z', we get the well known cubic equation

$$(e-F)(e-G)(e-H) - f^2(e-F) - g^2(e-G) - h^2(e-H)\ 2fgh = 0\ldots\ldots(4)$$

which occurs in the investigation of the principal axes of a rigid body, and in various others..........................

....

The three directions which have just been determined I shall call *axes of extension*. They will in general vary from one point to another, and from one instant of time to another. If we denote the three roots of (4) by e', e'', e''', and if we take new rectangular axes $Ox_{,}$, $Oy_{,}$, $Oz_{,}$, parallel to the axes of extension, and denote by $u_{,}$, $U_{,}$, &c. the quantities referred to these axes corresponding to u, U, &c., equations (3) must be satisfied by $y'_{,}=0$, $z'_{,}=0$, $e=e'$, by $x'_{,}=0$, $z'_{,}=0$, $e=e''$ and by $x'_{,}=0$, $y'_{,}=0$, $e=e'''$, which requires that $f_{,}=0$, $g_{,}=0$, $h_{,}=0$, and we have

$$e'=F_{,}=\frac{du_{,}}{dx_{,}},\quad e''=G_{,}=\frac{dv_{,}}{dy_{,}},\quad e''=H_{,}=\frac{dw_{,}}{dz_{,}}.$$

The values of $U_{,}$, $V_{,}$, $W_{,}$, which correspond to the residual motion after the elimination of the motion of rotation corresponding to ω', ω'' and ω''', are

$$U_{,}=e'x'_{,},\quad V_{,}=e''y'_{,},\quad W_{,}=e'''z'_{,}.$$

The angular velocity of which ω', ω'', ω''' are the components is independent of the arbitrary directions of the co-ordinate axes : the same is true of the directions of the axes of extension, and of the values of the roots of equation (4) . . .

....

The motion corresponding to the velocities $U_{,}$, $V_{,}$, $W_{,}$ may be further decomposed into a motion of dilatation, positive or negative, which is alike in all directions, and two motions which I shall call *motions of shifting*, each of the latter being in two dimensions, and not affecting the density. For let δ be the rate of linear extension corresponding to a uniform dilatation; let $\sigma x'$, $-\sigma y'$, be the velocities parallel to $x_{,}$, $y_{,}$, corresponding to a motion of shifting parallel to the plane $x_{,}y_{,}$, and let $\sigma' x'_{,}$, $-\sigma' z'$, be the velocities parallel to $x_{,}$, $z_{,}$, corresponding to a similar motion of shifting parallel to the plane $x_{,}z_{,}$. The velocities parallel to $x_{,}$, $y_{,}$, $z_{,}$ respectively corresponding to the quantities δ, σ and σ' will be $(\delta+\sigma+\sigma')\,x'_{,}$, $(\delta-\sigma)\,y'_{,}$, $(\delta-\sigma')\,z'_{,}$, and equating these to $U_{,}$, $V_{,}$, $W_{,}$ we shall get

$$\delta = \tfrac{1}{3}(e'+e''+e'''),\quad \sigma=\tfrac{1}{3}(e'+e'''-2e''),$$

$$\sigma'=\tfrac{1}{3}(e'+e''-2e''').$$

Hence the most general instantaneous motion of an elementary portion of a fluid is compounded of a motion of translation, a motion of rotation, a motion of uniform dilatation, and two motions of shifting of the kind just mentioned.

....

Art. 3. — ....

.... Let p be the pressure which would exist about the point P if the neighbouring molecules were in a state of relative equilibrium: let $p+p_{,}$ be the normal pressure, and $t_{,}$ the tangential action, both referred to a unit of surface, on a plane passing through P and having a given direction. By the hypotheses of Art. 1. the quantities $p_{,}$, $t_{,}$ will be independent of the angular velocities ω', ω'', ω''', depending only on the residual relative velocities U, V, W, or, which comes to the same, on e', e'' and e''', or on σ, σ' and δ. Since this residual motion is symmetrical with respect to the axes of extension, it follows that if the plane considered is perpendicular to any one of these axes the tangential action on it is zero, since there is no reason why it should act in one direction rather than in the opposite; for by the hypotheses of Art. 1. the change of density and temperature about the point P is to be neglected, the constitution of the fluid being ultimately uniform about that point. Denoting then by $p+p'$, $p+p''$, $p+p'''$ the pressures on planes perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$, we must have $p'=\phi(e', e'', e''')$, $p''=\phi(e'', e''', e')$, $p'''=\phi(e''', e', e'')$, $\phi(e', e'', e''')$ denoting a function of e', e'' and e''' which is symmetrical with respect to the two latter quantities. The question is now to determine, on whatever may seem the most probable hypothesis, the form of the function ϕ.

Let us first take the simpler case in which there is no dilatation, and only one motion of shifting, or in which $e''=-e'$, $e'''=0$, and let us consider what would take place if the fluid consisted of smooth molecules acting on each other by actual contact. On this supposition, it is clear, considering the magnitude of the pressures acting on the molecules compared with their masses, that they would be sensibly in a position of relative equilibrium, except when the equilibrium of any one of them became impossible from the displacement of the adjoining ones, in which case the molecule in question would start into a new position of equilibrium. This start would cause a corresponding displacement in the molecules immediately about the one which started, and this disturbance would be propagated immediately in all directions, the nature of the displacement however being different in different directions, and would soon become insensible. During the continuance of this disturbance, the pressure on a small plane drawn through the element considered would not be the same in all directions, nor normal to the plane: or, which comes to the same, we may suppose a uniform normal pressure p to act, together with a normal pressure $p_{,,}$, and a tangential force $t_{,,}$, $p_{,,}$ and $t_{,,}$ being forces of great intensity and short duration, that is being of the nature of impulsive forces. As the number of molecules comprised in the element considered has been supposed extremely great, we may take a time τ so short that all summations with respect to such intervals of time may be replaced without sensible error by integrations, and yet so long that a very great number of starts shall take place in it. Consequently we have only to consider the average effect of such starts and moreover we may without sensible error replace the impulsive forces such as $p_{,,}$ and $t_{,,}$ which succeed one another with great rapidity, by continuous forces. For planes perpendicular to the axes of extension these continuous will be the normal pressures p', p'', p'''.

Let us now consider a motion of shifting differing from the former only in having e' increased in the ratio of m to 1. Then if we suppose each start completed before the starts which would be sensibly affected by it are begun, it is clear that the same series of starts will take place in the second case as in the first, but at intervals of time which are less in the ratio of 1 to m. Consequently the continuous pressures by which the impulsive actions due to these starts may be replaced must be increased in the ratio of m to 1. Hence the pressures p', p'', p''' must be proportional to e', or we must have

$$p' = Ce', \quad p'' = C'e', \quad p''' = C''e'.$$

It is natural to suppose that these formulæ hold good for negative as well as positive values of e'. Assuming this to be true, let the sign of e' be changed. This comes to interchanging x and y, and consequently p''' must remain the same, and p' and p'' must be interchanged. We must therefore have $C'' = 0$, $C' = -C$. Putting then $C = -2\mu$ we have

$$p' = -2\mu e', \quad p'' = 2\mu e', \quad p''' = 0.$$

It has hitherto been supposed that the molecules of a fluid are in actual contact. We have every reason to suppose that this is not the case. But precisely the same reasoning will apply if they are separated by intervals as great as we please compared with their magnitudes, provided only we suppose the force of restitution called into play by a small displacement of *any one* molecule to be very great.

Let us now take the case of two motions of shifting which coexist, and let us suppose $e' = \sigma + \sigma'$, $e'' = -\sigma$, $e''' = -\sigma'$. Let the small time τ be divided into $2n$ equal portions, and let us suppose that in the first interval a shifting motion corresponding to $e' = 2\sigma$, $e'' = -2\sigma$ takes place parallel to the plane $x_{,}y_{,}$, and that in the second interval a shifting motion corresponding to $e' = 2\sigma'$, $e''' = -2\sigma'$ takes place parallel to the plane $x_{,}z_{,}$, and so on alternately. On this supposition it is clear that if we suppose the time $\tau/2n$ to be extremely small, the continuous forces by which the effect of the starts may be replaced will be $p' = -2\mu(\sigma + \sigma')$, $p'' = 2\mu\sigma$, $p''' = 2\mu\sigma'$. By supposing n indefinitely increased, we might make the motion considered approach as near as we please to that in which the two motions of shifting coexist; but we are not at liberty to do so, for in order to apply the above reasoning we must suppose the time $\tau/2n$ to be so large that the average effect of the starts which occur in it may be taken. Consequently it must be taken as an additional assumption, and not a matter of absolute demonstration, that the effects of the two motions of shifting are superimposed.

Hence if $\delta = 0$, *i.e.* if $e' + e'' + e''' = 0$, we shall have in general

$$p' = -2\mu e', \quad p'' = -2\mu e'', \quad p''' = -2\mu e''' \quad \ldots(5).$$

It was by this hypothesis of starts that I first arrived at these equations, and the differential equations of motion which result from them. On reading Poisson's memoir however, to which I shall have occasion to refer in Section IV., I was led to reflect that however intense we may suppose the molecular forces to be, and however near we may suppose the molecules to be to their positions of relative equilibrium, we are not therefore at liberty to suppose them *in* those positions, and consequently not at liberty to suppose the pressure equal in all directions in the intervals of time between the starts. In fact, by supposing the molecular forces indefinitely increased, retaining the same ratios to each other, we may suppose the displacements of the molecules from their positions of relative equilibrium indefinitely diminished, but on the other hand the force of restitution called into action by a given displacement is indefinitely increased in the same proportion. But be these displacements what they may, we know that the forces of restitution make equilibrium with forces equal and opposite to the effective forces; and in calculating the effective forces we may neglect the above displacements, or suppose the molecules to move in the paths in which they would move if the shifting motion took place with indefinite slowness. Let us first consider a single motion of shifting, or one for which $e'' = -e'$, $e''' = 0$, and let $p_{,}$ and $t_{,}$ denote the same quantities as before. If we now suppose e' increased in the ratio of m to 1, all the effective forces will be increased in that ratio, and consequently $p_{,}$ and $t_{,}$ will be increased in the same ratio. We may deduce the values of p' p'', and p''' just as before, and then pass by the same reasoning to the case of two motions of shifting which coexist, only that in this case the reasoning will be demonstrative, since we *may* suppose the time $\tau/2n$ indefinitely diminished. If we suppose the state of things considered in this paragraph to exist along with the motions of starting already considered, it is easy to see that the expressions for p', p'' and p''' will still retain the same form.

There remains yet to be considered the effect of the dilatation. Let us first suppose the dilatation to exist without any shifting: then it is easily seen that the relative motion of the fluid at the point considered is the same in all directions. Consequently the only effect which such a dilatation could have would be to introduce a normal pressure $p_{,,}$ alike in all directions, in addition to that due to the action of the molecules supposed to be in a state of relative equilibrium. Now the pressure $p_{,}$ could only arise from the aggregate of the molecular actions called into play by the displacements of the molecules from their positions of relative equilibrium; but since these displacements take place, on an average, indifferently in all directions, it follows that the actions of which $p_{,}$ is composed neutralize each other, so that $p_{,} = 0$. The same conclusion might be drawn from the hypothesis of starts, supposing, as it is natural to do, that each start calls into action as much increase of presssure in some directions as diminution of pressure in others.

If the motion of uniform dilatation coexists with two motions of shifting, I shall suppose, for the same reason as before, that the effects of these different motions are superimposed. Hence subtracting δ from each of the three quantities e', e'' and e''', and putting the remainders in the place of e', e'' and e''' in equations (5), we have

$$p' = \tfrac{2}{3}\mu(e'' + e''' - 2e'), \quad p'' = \tfrac{2}{3}\mu(e''' + e' - 2e''),$$
$$p''' = \tfrac{2}{3}\mu(e' + e'' - 2e''') \;.(6).$$

If we had started with assuming $\phi(e', e'', e''')$ to be a linear function of e', e'' and e''', avoiding all speculation as to the molecular constitution of a fluid, we should have had at once $p' = Ce' + C'(e'' + e''')$, since p' is symmetrical with respect to e'' and e'''; or, changing the constants, $p' = \tfrac{2}{3}\mu(e'' + e''' - 2e') + \kappa(e' + e'' + e''')$. The expressions for p'' and p''' would be obtained by interchanging the requisite quantities. Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same and in the third case nearly the same, whether κ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing κ to be equal to zero.

Art. 4. — It will be easy now to determine the oblique pressure, or resultant of the normal pressure and tangential action, on any plane. Let us first consider a plane drawn through the point P parallel to the plane yz. Let $Ox_{\prime}$ make with the axes of x, y, z angles whose cosines are l', m', n'; let l'',m'',n'' be the same for $Oy_{\prime}$, and l''', m''', n''' the same for $Oz_{\prime}$. Let P_1 be the pressure, and (xty), (xtz) the resolved parts, parallel to y, z respectively, of the tangential force on the plane considered, all referred to a unit of surface, (xty) being reckoned positive when the part of the fluid towards $-x$ urges that towards $+x$ in the positive direction of y, and similarly for (xtz). Consider the portion of the fluid comprised within a tetrahedron having its vertex in the point P, its base parallel to the plane yz, and its three sides parallel to the planes $x_{\prime}y_{\prime}$, $y_{\prime}z_{\prime}$, $z_{\prime}x_{\prime}$ respectively. Let A be the area of the base, and therefore $l'A$, $l''A$, $l'''A$ the areas of the faces perpendicular to the axes of $x_{\prime}$, $y_{\prime}$, $z_{\prime}$. By D'Alembert's principle, the pressures and tangential actions on the faces of this tetrahedron, the moving forces arising from the external attractions, not including the molecular forces, and forces equal and opposite to the effective moving forces will be in equilibrium, and therefore the sums of the resolved parts of these forces in the directions of x, y and z will each be zero. Suppose now the dimensions of the tetrahedron indefinitely diminished, then the resolved parts of the external, and of effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces on the sides as the squares of homologous lines. Dividing therefore the three equations arising from equating to zero the resolved parts of the above forces by A, and taking the limit, we have

$$P_1 = \Sigma l'^2 (p + p'), \quad (xty) = \Sigma l'm' (p + p'),$$
$$(xtz) = \Sigma l'n' (p + p'),$$

the sign Σ denoting the sum obtained by taking the quantities corresponding to the three axes of extension in succession. Putting for p', p'' and p''' their values given by (6), putting $e' + e'' + e''' = 3\,\delta$, and observing that $\Sigma l'^2 = 1$, $\Sigma l'm' = 0$, $\Sigma l'n' = 0$, the above equations become

$$P_1 = p - 2\,\mu \Sigma l'^2 e' + 2\,\mu\delta, \quad (xty) = -2\,\mu \Sigma l'm'e',$$
$$(xtz) = -2\,\mu \Sigma l'n'e'.$$

The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.

The most direct way of obtaining the values of $\Sigma l'^2 e'$ &c. would be to express l', m' and n' in terms of e' by any two of equations (3), in which x', y', z' are proportional to l', m', n', together with the equation $l'^2 + m'^2 + n'^2 = 1$, and then to express the resulting symmetrical function of the roots of the cubic equation (4) in terms of the coefficients. But this method would be excessively laborious, and need not be resorted to. For after eliminating the angular motion of the element of fluid considered the remaining velocities are $e'x'_{\prime}$, $e''y'_{\prime}$, $e'''z'_{\prime}$, parallel to the axes of $x_{\prime}$, $y_{\prime}$, $z_{\prime}$. The sum of the resolved parts of these parallel to the axis of x is $l'e'x'_{\prime} + l''e''y'_{\prime} + l'''e'''z'_{\prime}$. Putting for $x'_{\prime}$, $y'_{\prime}$, $z'_{\prime}$ their values $l'x' + m'y' + n'z'$ &c., the above sum becomes

$$x'\,\Sigma l'^2 e' + y' \Sigma l'm'e' + z' \Sigma l'n'e';$$

but this sum is the same thing as the velocity U in equation (2), and therefore we have

$$\Sigma l'^2 e' = \frac{du}{dx}, \quad \Sigma l'm'e' = \tfrac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right), \quad \Sigma l'n'e' = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)$$

It may also be very easily proved directly that the value of $3\,\delta$, the rate of cubical dilatation, satisfies the equation

$$3\,\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad \ldots\ldots\ldots\ldots (7).$$

Let P_2, (ytz), (ytx) be the quantities referring to the axis of y, and P_3, (ztx), (zty) those referring to the axis of z, which correspond to P_1 &c. referring to the axis of x. Then we see that $(ytz) = (zty)$, $(ztx) = (xtz)$, $(xty) = (ytx)$. Denoting these three quantities by T_1, T_2, T_3, and making the requisite substitutions and interchanges, we have

$$\left.\begin{aligned}
P_1 &= p - 2\,\mu\left(\frac{du}{dx} - \delta\right),\\
P_2 &= p - 2\,\mu\left(\frac{dv}{dy} - \delta\right),\\
P_3 &= p - 2\,\mu\left(\frac{dw}{dz} - \delta\right),\\
T_1 &= -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right),\\
T_2 &= -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right),\\
T_3 &= -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right),
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (8).$$

It may also be useful to know the components, parallel to x, y, z, of the oblique pressure on a plane passing through the point P, and having a given direction. Let l, m, n be the cosines of the angles which a normal to the given plane makes with the axes of x, y, z; let P, Q, R, be the components, referred to a unit of surface, of the oblique pressure on this plane, P, Q, R being reckoned positive when the part of the fluid in which is situated the normal to which l, m and n refer is urged by the other part in the positive directions of x, y, z, when l, m and n are positive. Then considering as before a tetrahedron of which the base is parallel to the given plane, the vertex in the point P, and the sides parallel to the co-ordinate planes, we shall have

$$\left.\begin{aligned}
P &= lP_1 + mT_3 + nT_2,\\
Q &= lT_3 + mP_2 + nT_1,\\
R &= lT_2 + mT_1 + nP_3,
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (9).$$

In the simple case of a sliding motion for which $u = 0$, $v = f(x)$, $w = 0$, the only forces, besides the pressure p, which act on planes parallel to the co-ordinate planes are the two tangential forces T_3, the value of which in this case is $-\mu\, dv/dx$. In this case it is easy to shew that the axes of extension are, one of them parallel to Oz, and the two others in a plane parallel to xy, and inclined at angles of 45° to Ox. We see also that it is necessary to suppose μ to be positive, since otherwise the tendency of the forces would be to increase the relative motion of the parts of the fluid, and the equilibrium of the fluid would be unstable.

Art. 5. — Having found the pressures about the point P on planes parallel to the co-ordinate planes, it will be easy to form the equations of motion. Let X, Y, Z be the resolved parts, parallel to the axes, of the external force, not including the molecular force; let ρ be the density, t the time. Consider an elementary parallelepiped of the fluid, formed by planes parallel to the co-ordinate planes, and drawn through the point (x, y, z) and the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. The mass of this element will be ultimately $\rho\, \Delta x \Delta y \Delta z$, and the moving force parallel to x arising from the external forces will be ultimately $\rho\, X \Delta x \Delta y \Delta z$; the effective moving force parallel to x will be ultimately $\rho\, Du/Dt . \Delta x \Delta y \Delta z$, where D is used, as it will be in the rest of this paper, to denote differentiation in which the independent variables are t and three parameters of the particle considered, (such for instance as its initial cordinates,) and not t, x, y, z. It is easy also to shew that the moving force acting on the element considered arising from the oblique pressures on the faces is ultimately

$$\left(\frac{dP}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) \Delta x \Delta y \Delta z,$$

acting in the negative direction. Hence we have by D'Alembert's principle

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \&c. \quad ..(10),$$

in which equations we must put for Du/Dt its value

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},$$

and similarly for Dv/dt and Dw/dt. In considering the general equations of motion it will be needless to write down more than one, since the other two may be at once derived from it by interchanging the requisite quantities. The equations (10), the ordinary equation of continuity, as it is called,

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0 \quad ....(11),$$

which expresses the condition that there is no generation or destruction of mass in the interior of a fluid, the equation connecting p and ρ, or in the case of an incompressible fluid the equivalent equation $D\rho/Dt = 0$, and the equation for the propagation of heat, if we choose to take account of that propagation, are the only equations to be satisfied at every point of the interior of the fluid mass.

As it is quite useless to consider cases of the utmost degree of generality, I shall suppose the fluid to be homogeneous, and of a uniform temperature throughout, except in so far as the temperature may be raised by sudden compression in the case of small vibrations. Hence in equations (10) μ may be supposed to be constant as far as regards the temperature; for, in the case of small vibrations, the terms introduced by supposing it to vary with the temperature would involve the square of the velocity, which is supposed to be neglected. If we suppose μ to be independent of the pressure also, and substitute in (10) the values of P_1 &c. given by (8), the former equations become

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \&c. \quad .....(12).$$

Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Dubuat, from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. The total retardation depends, partly on the friction of the water against the sides of the pipe or canal, and partly on the mutual friction, or tangential action, of the different portions of the water. Now if these two parts of the whole retardation were separately variable with p, it is very unlikely that they should when combined give a result independent of p. The amount of the internal friction of the water depends on the value of μ. I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure. On this supposition, we have from equations (11) and (12)

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0, \&c. \quad (13),$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect to friction on the motions of tides and waves, and such questions.

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Art. 6. — Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries. Let M be a point in the boundary of the fluid. Let a normal to the surface at M, draw on the outside of the fluid, make with the axes angles whose cosines are l, m, n. Let P', Q', R' be the components of the pressure of the fluid about M on the solid or fluid with which it is in contact, these quantities being reckoned positive when the fluid considered presses the solid or fluid outside it in the positive directions of x, y, z, supposing l, m and n positive. Let S be a very small element of the surface about M, which will be ultimately plane, S' a plane parallel and equal to S, and directly opposite to it, taken within the fluid. Let the distance between S and S' be supposed to vanish in the limit compared with the breadth of S, a supposition which may be made if we neglect the effect of the curvature of the surface at M; and let us consider the forces acting on the element of fluid comprised between S and S', and the motion of this element. If we suppose equations (8) to hold good to within an insensible distance from the surface of the fluid, we shall evidently have forces ultimately equal to PS, QS, RS, (P, Q and R being given by equations (9),) acting on the inner sdie of the element in the positive directions of the axes, and forces ultimately equal to $P'S$, $Q'S$, $R'S$ acting on the outer side in the negative directions. The moving forces arising from the external forces acting on the element, and the effective moving forces will vanish in the limit compared with the forces PS, &c.: the same will be true of the pressures acting about the edge of the element, if we neglect capillary attraction, and all forces of the same nature. Hence, taking the limit, we shall have

$$P' = P, \quad Q' = Q, \quad R' = R.$$

The method of proceeding will be different according as the bounding surface considered is a free surface, the surface of a solid, or the surface separation of two fluids, and it will be necessary to consider these cases separately. Of course the surface of a liquid exposed to the air is really the surface of separation of two fluids, but it may in many cases be regarded as a free surface if we neglect the inertia of the air: it may always be so regarded if we neglect the friction of the air as well as its inertia.

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Art. 9. — Although the discharge of a liquid through a long straight pipe or canal, under given circumstances, cannot be calculated without knowing the conditions to be satisfied at the surface of contact of the fluid and solid, it may be well to go a certain way towards the solution.

Let the axis of z be parallel to the generating lines of the pipe or canal, and inclined at an angle α to the horizon; let the plane yz be vertical, and let y and z be measured downwards.

The motion being uniform, we shall have $u = 0$, $v = 0$, $w = f(x, y)$, and we have from equations (13)

$$\frac{dp}{dx} = 0,\ \frac{dp}{dy} = g\rho \cos \alpha,\ \frac{dp}{dz} = g\rho \sin \alpha + \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right).$$

In the case of a canal $dp/dz = 0$; and the calculation of the motion in a pipe may always be reduced to that of the motion in the same pipe when dp/dz is supposed to be zero, as may be shewn by reasoning similar to Dubuat's. Moreover the motion in a canal is a particular case of the motion in a pipe. For consider a pipe for which $dp/dz = 0$, and which is divided symmetrically by the plane xz. From the symmetry of the motion, it is clear that we must have $dw/dy = 0$ when $z = 0$; but this is precisely the condition which would have to be satisfied if the fluid had a free surface coinciding with the plane xz; hence we may suppose the upper half of the fluid removed, without affecting the motion of the rest, and thus we pass to the case of a canal. Hence it is the same thing to determine the motion in a canal, as to determine that in the pipe formed by completing the canal symmetrically with respect to the surface of the fluid.

We have then, to determine the motion, the equation

$$\frac{d^2w}{dx} + \frac{d^2w}{dy^2} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

In the case of a rectangular pipe, it would not be difficult to express the value of w at any point in terms of its values at the several points of the perimeter of a section of the pipe. In the case of a cylindrical pipe the solution is extremely easy: for if we take the axis of the pipe for that of z, and take polar coordinates r, θ in a plane parallel to xy, and observe that $dw/d\theta = 0$, since the motion is supposed to be symmetrical with respect to the axis, the above equation becomes

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

Let a be the radius of the pipe, and U the velocity of the fluid close to the surface; then, integrating the above equation, and determining the arbitrary constants by the conditions that w shall be finite when $r = 0$, and $w = U$ when $r = a$, we have

$$w = \frac{g\rho \sin \alpha}{4\mu}(a^2 - r^2) + U.$$

. . . .

Section II.

Objections to Lagrange's proof of the theorem that if udx + vdy + wdz *is an exact differential at any one instant it is always so, the pressure being supposed equal in all directions. Principles of M. Cauchy's proof. A new proof of the theorem. A physical interpretation of the circumstance of the above expression being an exact differential.*

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Section III.

Application of a method analogous to that of Sect. I. *to the determination of the equations of equilibrium and motion of elastic solids.*

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Section IV.

Principles of Poisson's theory of elastic solids, and of the oblique pressures existing in fluids in motion. Objections to one of his hypotheses. Reflections on the constitution, and equations of motion of the luminiferous ether in vacuum.

Appendix

REPORT ON RECENT RESEARCHES IN HYDRODYNAMICS.

by G. G. Stokes.

VI. M. Navier was, I believe, the first to give equations for the motion of fluids without supposing the pressure equal in all directions. His theory is contained in a memoir read before the French Academy in 1822 [*]. He considers the case of a homogeneous incompressible fluid. He supposes such a fluid to be made up of ultimate molecules, acting on each other by forces which, when the molecules are at rest, are functions simply of the distance, but which, when the molecules recede from, or approach to each other, are modified by this circumstance, so that two molecules repel each other less strongly when they are receding, and more strongly when they are approaching, than they do when they are at rest [†]. Then alteration of attraction or repulsion is supposed to be, for a given distance, proportional to the velocity with which the molecules recede from, or approach to each other; so that the mutual repulsion of two molecules will be represented by $f(r) - VF(r)$, where r is the distance of the molecules, V the velocity with which they recede from each other, and $f(r)$, $F(r)$ two unknown functions of r depending on the molecular force, and as such becoming insensible when r has become sensible. This expression does not suppose the molecules to be necessarily receding from each other, nor their mutual action to be necessarily repulsive, since V and $F(r)$ may be positive or negative. It is not absolutely necessary that $f(r)$ and $F(r)$ should always have the same sign. In forming the equations of motion M. Navier adopts the hypothesis of a symmetrical arrangement of the particles, or at least, which leads to the same result, neglects the irregular part of the mutual action of neighbouring molecules. The equations at which he arrives are those which would be obtained from the common equations by writing $\frac{dp}{dx} - A\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{du^2}{dz^2}\right)$ in place of $\frac{dp}{dx}$ in the first, and making similar changes in the second and third. A is here an unknown constant depending on the nature of the fluid.

The same subject has been treated on by Poisson [‡], who has adopted hypotheses which are very different from those of M. Navier. Poisson's theory is of this nature. He supposes the time t to be divided into n equal parts, each equal to τ. In the first of these he supposes the fluid to be displaced in the same manner as an elastic solid, so that the pressures in different directions are given by the equations which he had previously obtained for elastic solids. If the causes producing the displacement were now to cease to act, the molecules would very rapidly assume a new arrangement, which would render the pressure equal in all directions, and while this re-arrangement was going on, the pressure would alter in an unknown manner from that belonging to a displaced elastic solid to the pressure belonging to the fluid in its new state. The causes of displacement are however going on during the second interval τ; but since these different small motions will take place independently, the new displacement which will take place in the second interval τ will be the same as if the molecules were not undergoing a re-arrangement. Supposing now n to become infinite, we pass to the case in which the fluid in continually beginning to be displaced like an elastic solid, and continually re-arranging itself so as to make the pressure equal in all directions. The equations at which Poisson arrived are, in the cases of a homogeneous incompressible fluid, and of an elastic fluid in which the change of density is small,

[*] *Mémoires de l'Académie des Sciences*, tom. vi. p. 389.
[†] This idea appears to have been borrowed from Dubuat. See his *Principes d'Hydraulique*, tom. ii. p. 60.
[‡] *Journal de l'Ecole Polytechnique*, tom. xiii. cah. 20, p. 139.

those which would be derived from the common equations by replacing dp/dx in the first by

$$\frac{dp}{dx} - A\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - B\,\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

and making similar changes in the second and third. In these equations A and B are two unknown constants. It will be observed that Poisson's equations reduce themselves to Navier's in the case of an incompressible fluid.

The same subject has been considered in a quite different point of view by M. Barré de Saint-Venant, in a communication to the French Academy in 1843, an abstract of which is contained in the *Comptes Rendus* *. The principal difficulty is to connect the oblique pressures in different directions about the same point with the differential coefficients du/dx, du/dy, &c., which express the relative motion of the fluid particles in the immediate neighbourhood of that point. This the author accomplishes by assuming that the tangential force on any plane passing through the point in question is in the direction of the principal sliding (*glissement*) along that plane. The sliding along the plane xy is measured by $\frac{dw}{dx} + \frac{du}{dz}$ in the direction of x, and $\frac{dw}{dy} + \frac{dv}{dz}$ in the direction of y. These two slidings may be compounded into one, which will form the principal sliding along the plane xy. It is then shewn, by means of M. Cauchy's theorems connecting the pressures in different directions in any medium, that the tangential force on any plane through the point considered, resolved in any direction in that plane, is proportional to the sliding along that plane resolved in the same direction, so that if T represents the tangential force, referred to a unit of surface, and S the sliding, $T = \epsilon S$. The pressure on a plane in any direction is then found. This pressure is compounded of a normal pressure, alike in all directions, and a variable oblique pressure, the expression for which contains the one unknown quantity ϵ. If the fluid be supposed incompressible, and ϵ constant, the equations which would be obtained by the method of M. Barré de Saint-Venant agree with those of M. Navier. It will be observed that this method does not require the consideration of ultimate molecules at all.

When the motion of the fluid is very small, Poisson's equations agree with those given by M. Cauchy for the motion of a solid entirely destitute of elasticity †, except that the latter do not contain the pressure p. These equations have been obtained by M. Cauchy without the consideration of molecules. His method would apply, with very little change, to the case of fluids.

In a paper read last year before the Cambridge Philosophical Society ‡, I have arrived at the equations of motion in a different manner. The method employed in this paper does not necessarily require the consideration of ultimate molecules. Its principal feature consists in eliminating from the relative motion of the fluid about any particular point the relative motion which corresponds to a certain motion of rotation, and examining the nature of the relative motion which remains. The equations finally adopted in the cases of a homogeneous incompressible fluid, and of an elastic fluid in which the change of density is small, agree with those of Poisson, provided we suppose in the latter $A = 3B$. It is shewn that this relation between A and B may be obtained on Poisson's own principles.

The equations hitherto considered are those which must be satisfied at any point in the interior of the fluid mass; but there is hardly any instance of the practical application of the equations, in which we do not want to know also the particular conditions which must be satisfied at the surface of the fluid. With respect to a free surface there can be little doubt: the condition is simply that there shall be not tangential force on a plane parallel to the surface, taken immediately within the fluid. As to the case of a fluid in contact with a solid, the condition at which Navier arrived comes to this: that if we conceive a small plane drawn within the fluid parallel to the surface of the solid, the tangential force on this plane, referred to a unit of surface, shall be in the same direction with, and proportional to the velocity with which the fluid flows past the surface of the solid. The condition obtained by Poisson is essentially the same.

Dubuat stated, as a result of his experiments, that when the velocity of water flowing through a pipe is less than a certai[n] quantity, the water adjacent to the surface of the pipe is [at] rest *. This result agrees very well with an experiment of Coulomb's. Coulomb found that when a metallic disc was made to oscillate very slowly in water about an axis passing through its centre and perpendicular to its plane, the resistance was not altered when the disc was smeared with grease; and even when the grease was covered with powdered sandstone the resistance was hardly increased †. This is just what one would expect on the supposition that the water close to the disc is carried along with it, since in that case the resistance must depend on the internal friction of the fluid; but the result appears very extraordinary on the supposition that the fluid in contact with the dics flows past it with a finite velocity. It should be observed, however, that this result is compatible with the supposition that a thin film of fluid remains adhering to the disc, in consequence of capillary attraction, and becomes as it were solid, and that the fluid in contact with this film flows past it with a finite velocity. If we consider Dubuat's supposition to be correct, the condition to be assumed in the case of a fluid in contact with a solid is that the fluid does not move relatively to the solid. This condition will be included in M. Navier's, if we suppose the coefficient of the velocity when M. Navier's condition is expressed analytically, which he denotes by E, to become infinite. It seems probable from the experiments of M. Girard, that the condition to be satisfied at the surface of fluid in contact with a solid is different according as the fluid does or not moisten the surface of the solid.

M. Navier has applied his theory to the results of some experiments of M. Girard's on the discharge of fluids through capillary tubes. His theory shews that if we suppose E to be finite, the discharge through extremely small tubes will depend only on E, and not on A. The law of discharge at which he arrives agrees with the experiments of M. Girard, at least when the tubes are extremely small. M. Navier explained the difference observed by M. Girard in the discharge of water through tubes of glass and tubes of copper of same size by supposing the value of E different in the two cases. This difference was explained by M. Girard himself by supposing that a thin film of fluid remains adherent to the pipe, in consequence of molecular action, and that the thickness of this film differs with the substance of which the tube is composed, as well as with the liquid employed ‡. If we adopt Navier's explanation, we may reconcile it with the experiments of Coulomb by supposing that E is very large, so that unless the fluid is confined in a very narrow pipe, the results will depend mainly on A, being sensibly the same as they would be if E were infinite.

* Tom. xvii. p. 1240.
† *Exercices de Mathématiques*, tom. iii., p. 187.
‡ *Transactions of the Cambridge Philosophical Society*, vol. viii. p. 287.

* See the Table given in tom. i. of his *Principes d'Hydraulique*, p. 93.
† *Mémoires de l'Institut*, 1801, tom. iii. p. 286.
‡ *Mémoires de l'Académie des Sciences*, tom. i. pp. 203 and 234.