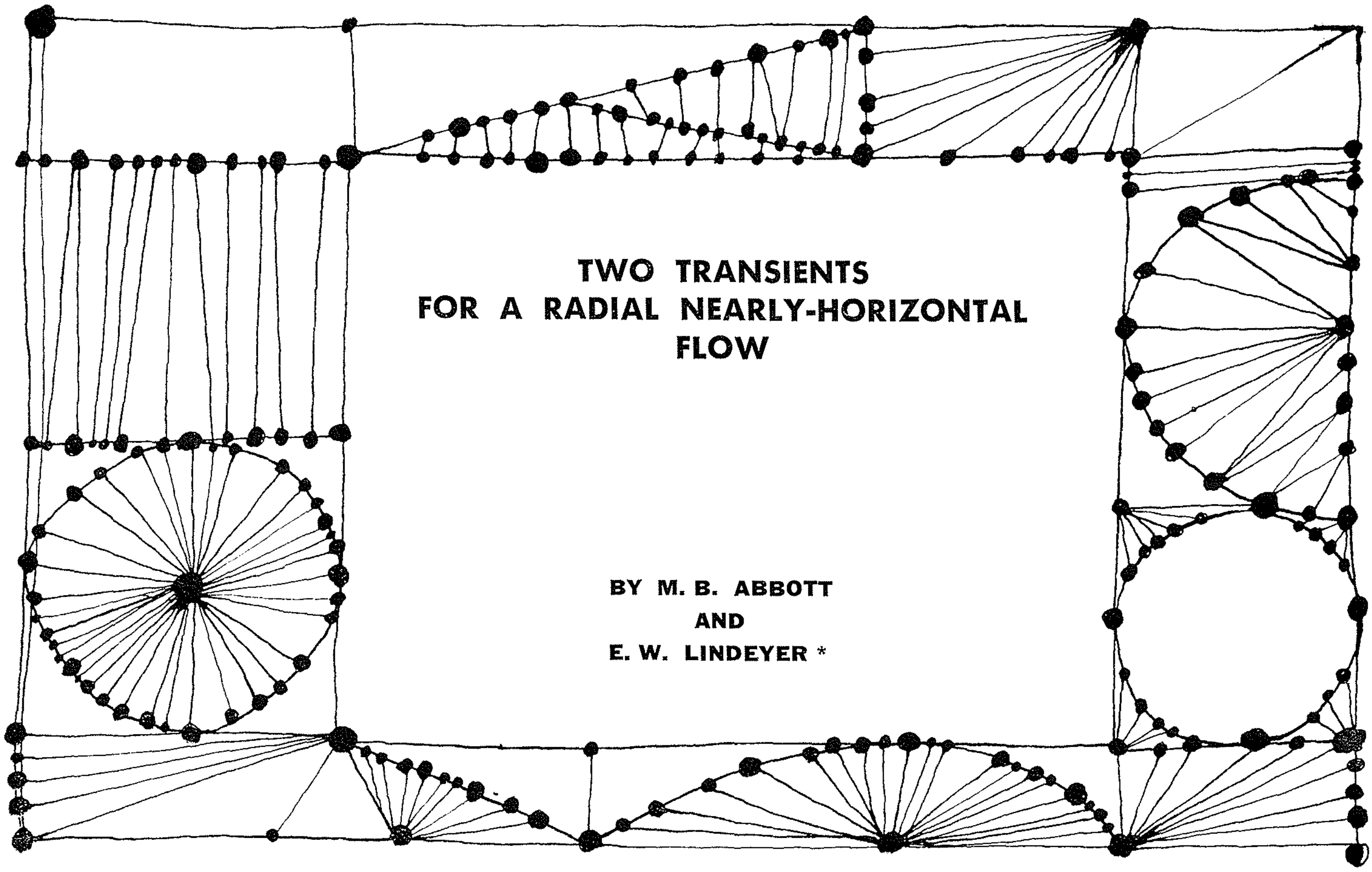

TWO TRANSIENTS FOR A RADIAL NEARLY-HORIZONTAL FLOW

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Introduction

By a transient we understand a sequence of states connecting one steady state of a system to another. A wide class of transients are associated with the following problem:

"In an n-dimensional Euclidean space E_n a function f (which may be a vector function) is zero everywhere for all times $t < t_0$. For all times $t \geqslant t_0$ a value f_0 of f is imposed in a domain X_0 in E_n (which may be a single point in E_n). Determine $f = f(x, t)$ for all x in E_n and all $t \geqslant t_0$."

For $n = 1$ and f taken as the elevation of fluid in a perfectly linear nearly-horizontal flow with propagation celerity c, the solution is well known:

$$f = f_0 . \mathrm{H}(ct - |x|) \qquad t \geqslant 0$$

where f_0 is the elevation imposed at the "origin" where $x = x_0 = 0$, and $\mathrm{H} = \mathrm{H}(a)$ is the Heaviside step function defined as:

$$\mathrm{H}(a) = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a \geqslant 0. \end{cases}$$

When viewed as a purely initial-value problem this solution coincides with the "fundamental solution" of the perfectly linear wave equation $f_{tt} - c^2 f_{xx} = 0$, being generated from initial conditions:

$$f(x, 0) = 0$$

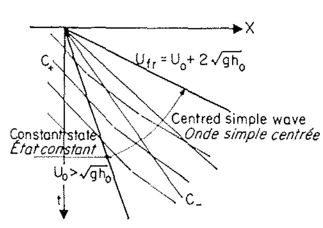
$$f_t(x, 0) = 2 f_0 c\, \delta(x)$$

where $\delta(x)$ is the Dirac "δ-function", new commonly called the δ-measure or δ-distribution (e.g. Marchand, 1962). The final "equilibrium" state (at $t = \infty$) corresponds to a function that is constant at f_0 for all finite x.

For $n = 1$ and $f = \{h, u\}$, the vector formed from depth and velocity of a quasi-linear nearly-horizontal flow, we find that no solution exists for $|u_0| < \sqrt{gh_0}$, while for $|u_0| \geqslant \sqrt{gh_0}$ a unique solution exists only on one side of the x-axis. This solution can be schematized as a centred simple wave, as shown for $x > 0$ in Figure 1 (Abbott, 1966).

The state of the system at $t = \infty$ is an equilibrium state with $u = u_0$ and $h = h_0$ for all finite x. However, as for all $t < \infty$, this solution implies that the Froude number $\mathfrak{F} \rightarrow \infty$ towards the front, we suppose that it is not physically realistic towards the front.

In this note we shall give the solution for $n = 2$ and $f = \{h, u\}$, the vector formed from the depth and radial velocity of quasi-linear nearly-horizontal flow. However, as this transient-solution will again lead to a front with $\mathfrak{F} \rightarrow \infty$, we shall also consider one other form of front condition, that in fact provides a quite different transient.



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Although many other transients are to be found in the books on Heat Conduction and Hydrodynamics (e.g. Carslaw and Jaeger, 1959, Lamb, 1932), the two solutions constructed here do not appear to have been given previously.

Systems of conservation laws and the law of the front

The laws of conservation of mass and momentum for a perfectly radial nearly-horizontal flow can be written in primitive form as:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} + h\frac{\partial u}{\partial r} + \frac{uh}{r} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + g\frac{\partial h}{\partial r} = 0 \qquad (2)$$

(e.g. Abbott, 1966). The same equations appear for an uncoupled surface layer in a stratified fluid, with the exception that $(1-\lambda)\,g$ appears instead of g. In this latter case, λ is the ratio of the density of the uncoupled layer to the layer upon which it is superposed, and $\lambda < 1$ for stability. Equation (1) can be written in conservation form (Lax, 1954) as:

$$\frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}(ruh) = 0 \qquad (3)$$

but the momentum equation only reduces to:

$$\frac{\partial}{\partial t}(uhr) + \frac{\partial}{\partial r}(u^2hr) + r\frac{\partial}{\partial r}(gh^2/2) = 0 \qquad (4)$$

due to the presence of impulses from the fluid "walls" surrounding any radial element. On the other hand, the energy equation does reduce to conservation form:

$$\frac{\partial}{\partial t}\{rh\,(u^2/2 + gh/2)\} + \frac{\partial}{\partial r}\{ruh\,(u^2/2 + gh)\} = 0 \qquad (5)$$

The quasi-linear system (1, 2) defines characteristics $\dot{x} = u \pm \sqrt{gh}$ and corresponding pseudo-invariants:

$$\Big[u \pm 2\sqrt{gh}\Big]_1^2 = \mp \int_{t_1}^{t_2} \frac{u\sqrt{gh}}{r}\,dt, \qquad (6)$$

where the bracket denotes a difference between the enclosed quantity between characteristically connected states 1 and 2.

In the case of a time-steady flow, (3) and (5) imply that:

$$uhr = \text{constant}$$
$$\frac{u^2}{2} + gh = \text{constant} \qquad (7)$$

throughout the flow. The second of these constitutes a strong form of Bernoulli's law.

At a discontinuity, realised as a hydraulic jump, the second law of thermodynamics insists that we take mass and momentum equations to form a genuine system of conservation laws. We ought then to obtain, on the assumption that flows are nearly-horizontal immediately to one side and the other of the discontinuity, that:

$$c\begin{bmatrix} h \\ uh \end{bmatrix}_1^2 = \begin{bmatrix} uh \\ u^2h + gh^2/2 \end{bmatrix}_1^2 \qquad (8)$$

where the brackets denote differences in the enclosed vector from side 1 to side 2 of the discontinuity. We remark that although the first equality in (8) is consistent with the continuous form (3), the second equality is not consistent with (4).

We shall consider two types of front condition. In the first place we consider a St. Venant front, identified by a coincidence of two characteristics of the same sense (Abbott and Torbe, 1963; Boulot, Braconnot and Marvaud, 1967). In the second case we consider a front defined by a Jeffreys-Vedernikov stability condition (Engelund, 1965), whereby the increase in resistance associated with this criterion is supposed sufficient to maintain the front slope, (Abbott, 1965, 1966). We have illustrated the St. Venant front for a rectilinear flow in Figure 1, and to this we now add Figure 2, showing possible Jeffreys-Vedernikov fronts. For $\mathcal{F}_0 < 2$ we obtain the characteristic structure shown in Figure 2 *a*, while for $\mathcal{F}_0 > 2$ we obtain two regions of constant state separated by a hydraulic jump, as shown in Figure 2 *b*.

We see from the characteristic structures and associated profiles of Figure 2 that a change in type of flow occurs as u passes through $2\sqrt{gh}$. This change of type will clearly occur whenever we characterize a front by a Froude number.

Two theorems for steady radial flows

We shall first prove the following theorem:

I. — In any radial flow that is nearly-horizontal and time-steady in some domain D we find that :

i) If at some radius r_1 in D the Froude number $\mathcal{F}_1 < 1$, then $\mathcal{F}$ decreases monotonically in r throughout D;

ii) If at some radius r_1 in D the Froude number $\mathcal{F}_1 > 1$, then $\mathcal{F}$ increases monotonically in r throughout D.

In fact, in any such steady flow we shall have:

$$uhr = \text{constant} = \alpha, \text{ say}$$
$$\frac{u^2}{2g} + h = \text{constant} = \beta, \text{ say} \qquad (9)$$

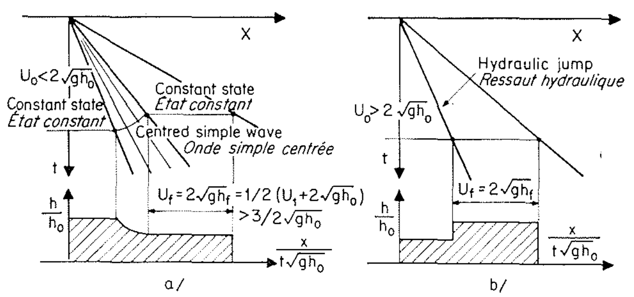
for all r in D. We use these equalities to compare conditions at the given radius r_1 and at some other radius $r_2 > r_1$, employing the usual plot of Specific Energy against depth (Fig. 3). In this plot, every point defines a unique value of the vector $\{u, h\}$. Now the condition $r_2 > r_1$ implies that $u_2h_2 < u_1h_1$ so that for any given h, $u_2 < u_1$. Thus the graph of $u^2/2\,g$ against h for r_2 lies below that of $u^2/2\,g$ against h for r_1 in Figure 3, and thence, in turn, the graph of $(u^2/2\,g + h)$ against h for r_2 always lies below that of $(u^2/2\,g + h)$ against h for r_1. We next consider the ensemble of all curves corresponding to radii $r_1 < r_2 < r_3 <$ etc. and the line $u^2/2g + h = \beta$ parallel to the h-axis, as indicated in Figure 3 *b*. We see that as r increases so the point defining the corresponding $\{u, h\}$ moves along $u^2/2g + h = \beta$ such that, if $\mathcal{F} < 1$ it moves to the right with increasing r, while if $\mathcal{F} > 1$ it moves to the left with increasing r.

If now, we introduce lines of equal Froude number in Figure 3 *b*, we see that as the representative point moves to the right, $\mathcal{F}$ decreases monotonically, while as it moves to the left, $\mathcal{F}$ increases monotonically. Thus, for $\mathcal{F} < 1$ at any radius, $\mathcal{F}$ decreases with increasing radius, while for $\mathcal{F} > 1$ at any radius, $\mathcal{F}$ increases with increasing radius. As we move inwards, i.e. as r is reduced, so the representative point moves in either case towards the critical condition, which thus appears as a boundary to the domain D. Conversely, a critical flow at some radius r_c will either provide a flow with $\mathcal{F} > 1$ at all $r > r_c$, or it will provide a flow with $\mathcal{F} < 1$ at all $r > r_c$. These two developments are schematized as characteristic structures in Figure 4.

Theorem I leads to theorem II.

II. — If a steady radial flow is nearly-horizontal and critical at some radius r_c, then there exists some $\varepsilon > 0$ such that it is not nearly-horizontal in the range $[r_c - \varepsilon, r_c]$.

For suppose that the flow is nearly-horizontal in the range $[r_c - \varepsilon, r_c]$ for some $\varepsilon > 0$. Then either $\mathcal{F} < 1$, or $\mathcal{F} = 1$ or $\mathcal{F} > 1$ at some $(r_c - \varepsilon) < r < r_c$. However, if $\mathcal{F} < 1$ at $r < r_c$ then, by I, $\mathcal{F} < 1$ at r_c, and if $\mathcal{F} = 1$ at r_c then either $\mathcal{F} < 1$ or $\mathcal{F} > 1$ at r_c and if $\mathcal{F} > 1$ at $r < r_c$ then $\mathcal{F} > 1$ at r_c. Since we have $\mathcal{F} = 1$ at r_c, all possible values of $\mathcal{F}$ at r within the range $[r_c - \varepsilon, r_c]$ lead to a contradiction for $\varepsilon > 0$. Thus the flow cannot be nearly-horizontal throughout the range $[r_c - \varepsilon, r_c]$ for any $\varepsilon > 0$. There remains the possibility, however, that it is not nearly-horizontal at the point r_c itself, but nearly-horizontal elsewhere in the range. In this case, however, a discontinuity would have to occur at r_c. However, since the flow is steady and critical at r_c any such discontinuity could only connect the state at $r_c - 0$ to the same state at $r_c + 0$ and thus could not be any true discontinuity at all. Thus



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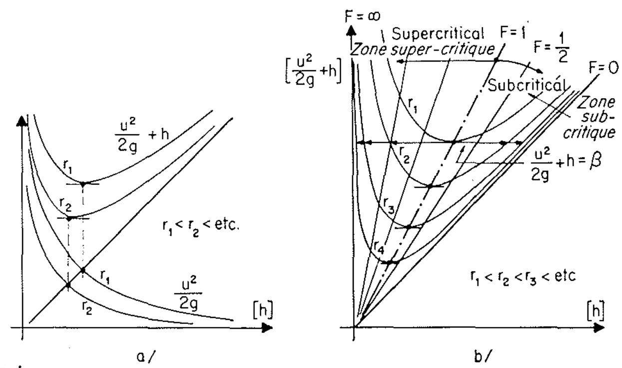

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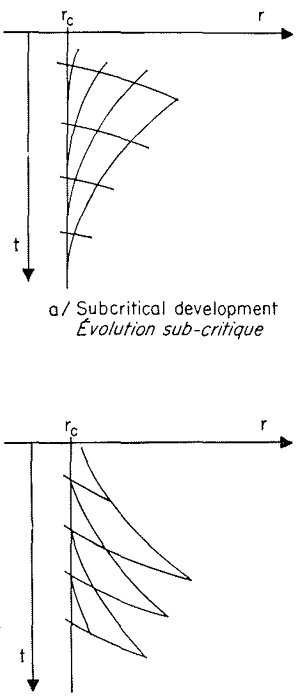

a/ Subcritical development
Évolution sub-critique

b/ Supercritical development
Évolution super-critique

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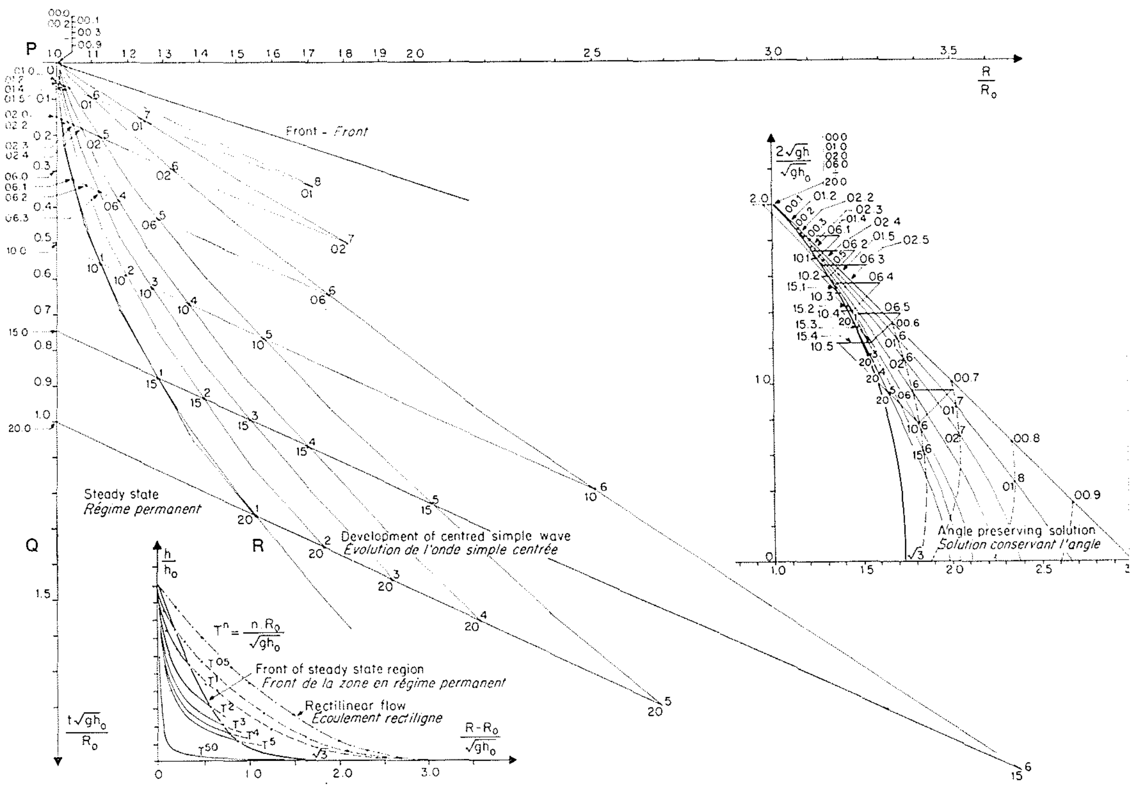

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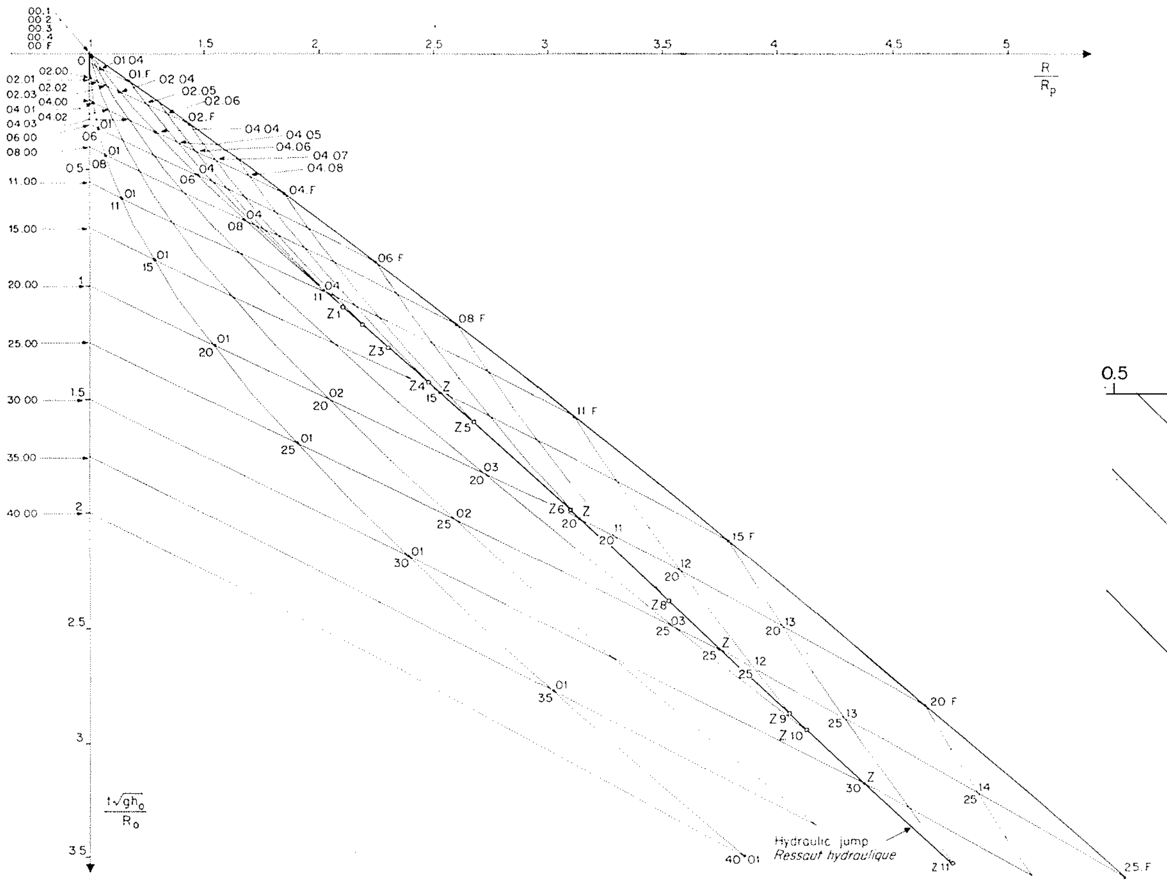

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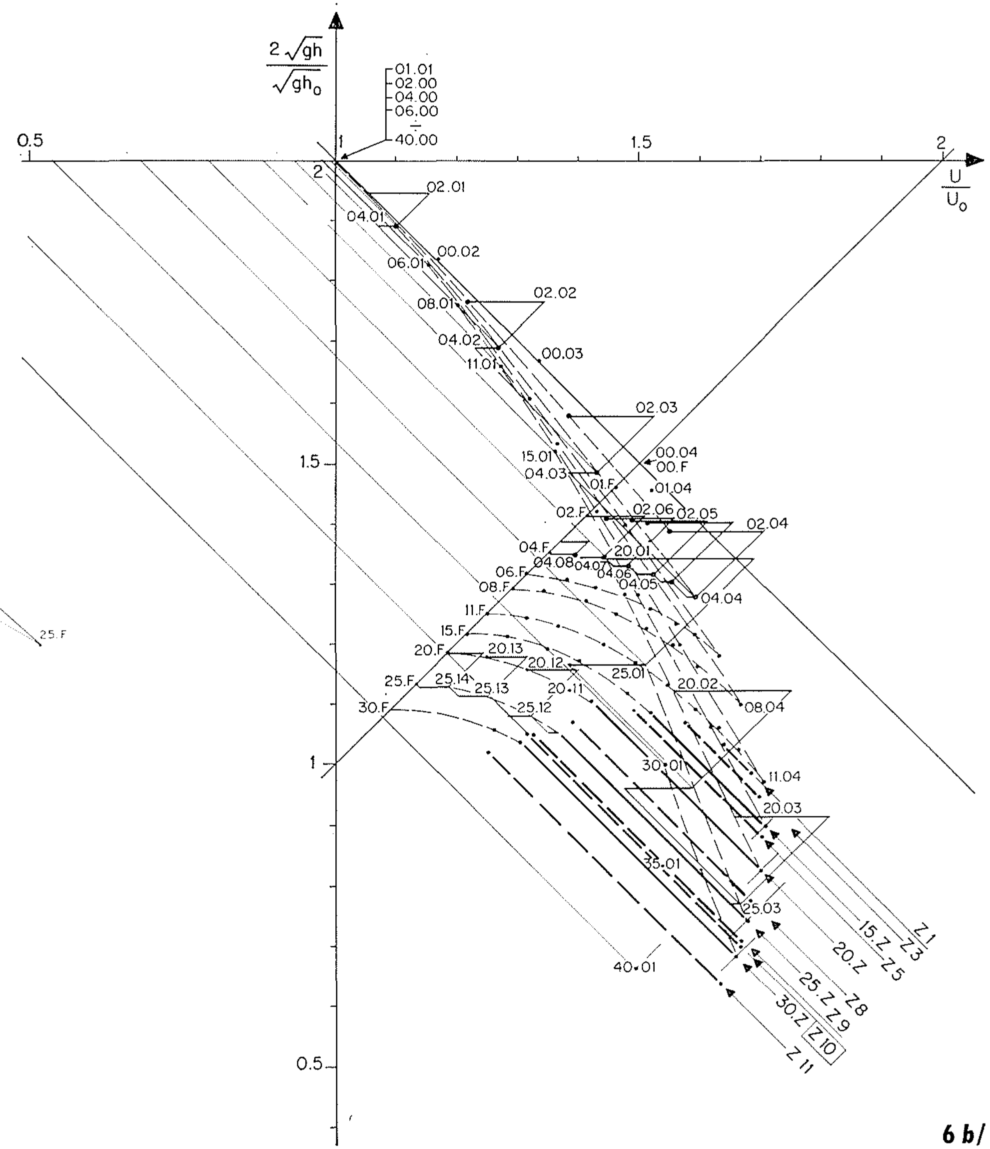


6 b/

there exists a finite ε such that the flow is not nearly-horizontal within the range $[r_c - \varepsilon, r_c]$. It follows from the above theorem that any nearly-horizontal radial flow with a supercritical front must have either a supercritical or a critical internal boundary condition at radius r_0. In the latter case the flow may have a continuation for $r < r_0$ but this must then be described by a higher order theory, in which vertical accelerations are introduced (e.g. the second-order theory enunciated by Boussinesq and developed by Massé; Jaeger, 1957). We associate this latter situation with that occurring in "free flows", in which fluid is supposed to leave or enter a system with minimum energy flux for given mass flux, which criterion corresponds, of course, to $\mathcal{F} = 1$ in the present case.

In our formulation of the transient problem we shall suppose that internal boundary conditions correspond to a free flow, or critical flow. Thus, if an energy flux β and mass flux α are given, a definite radius r_0 is defined, at which the flux becomes critical and nearly-horizontal:

$$r_0 = g\,\alpha \left(\frac{2\,g\beta}{3}\right)^{-3/2} \qquad (10)$$

The transient for a critical inflow and St. Venant front

The transient has been constructed using the quasi-invariants (6) and the steady flow relations (7) from a boundary radius defined by (10). Initially, at point P (Fig. 5), the flow behaves as a rectilinear flow (Fig. 1), but with increasing time a steady state region is formed adjacent to the inner boundary PQ. The flow accelerates outwards across this steady-state region as far as the characteristic PR. Along PR we have $uhr = u_0 h_0 r_0$, while the quantity of fluid passing any circumference through R travelling outwards with the corresponding characteristic celerity is, in unit time, $2\,\pi r \sqrt{gh}\,h$. However, $\sqrt{gh}$ at R rapidly becomes very much less than u, so that, as time progresses, an ever greater part of the total mass that has entered the system is to be found in the steady-state region. Thence we conclude that the transient develops into a time steady solution defined by (5), followed by a thin sheet of fluid that, in the limit $t \to \infty$, approaches an ε-distribution (Abbott, 1966).

The front of the steady-state solution starts with celerity zero and approaches a celerity $\sqrt{3\,gh_0}$ assymptotically. From a physical point of view, such a solution is, of course, exceedingly unrealistic.

The transient for a critical inflow and a Jeffreys-Vedernikov front

The physical and hodograph planes for this case, constructed using the quasi-invariants (6), are shown in Figure 6 *a* and 6 *b*. From these planes we have derived depth profiles for:

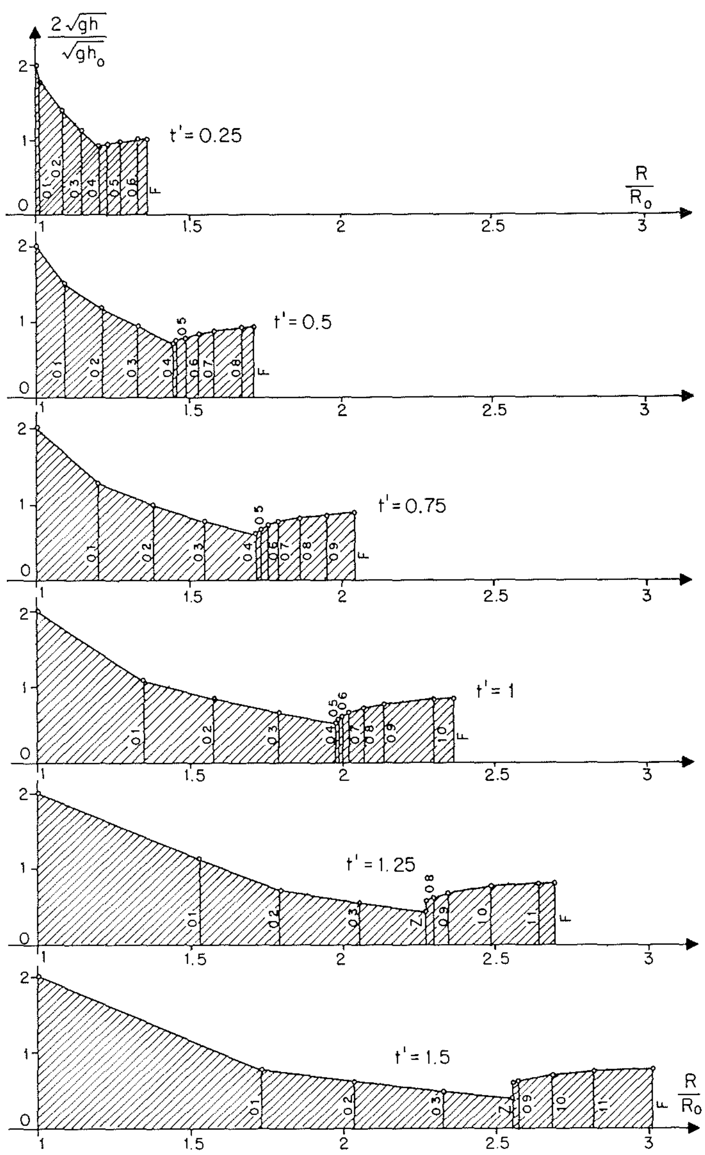
$$0 < t\sqrt{gh_0}/r_0 \leqslant 1.5,$$

as shown in Figure 7. We remark the formation of a hydraulic jump in this case, as a consequence of the outward acceleration of the fluid on the one hand and the Froude number formulation of the front condition on the other.

Conclusions

We have formulated the transient problem precisely for radial flows in the following way:

"For all $t < t_0$ the depth of fluid over the horizontal plane is everywhere zero, so that the velocity u is everywhere undefined. For all times $t \geqslant t_0$ a value h_0 of h and a value u_0 of u are imposed at a boundary radius r_0, so that there exists a constant α satisfying the relation $u_0^2 = \alpha\, gh_0$. Determine $h = h(r, t)$ and $u = u(r, t)$ for all $r > r_0$ and all $t \geqslant t_0$."



7/ Depth profiles for $0 < t\sqrt{gh_0}/R_0 = t' \leqslant 1.5$.
Profils de profondeur correspondant à $0 < t\sqrt{gh_0}/R_0 = t' \leqslant 1{,}5$.

We have distinguished two situations:

i) When $\alpha < 1$ no transient can exist, in that $\hat{C}$ characteristics enter the boundary from the front and impose a relation between u_0 and h_0 which is not of the form $u_0^2 = \alpha g h_0$ with α constant for any $t > t_c$;

ii) When $\alpha \geqslant 1$ the problem is correctly posed in that, for each value of α it has a unique solution. In the case that $\alpha > 1$ the flow at r_0 has a spatial continuation for some $r < r_0$ where flow remains nearly-horizontal, but in the case that $\alpha = 1$ no such continuation is possible. This latter case corresponds to a free discharge, where energy flux is a minimum for the given mass flux.

In order that our formulation should be unique and physically significant, we therefore introduce the condition that $\alpha = 1$.

The solution of the problem when flow is nearly-horizontal everywhere, i.e. with a St. Venant front, is then found to consist of a region of steady state, growing outwards from the boundary in time, followed by a development of a centred simple wave. As time progresses, the steady state region comes to contain a greater and greater part of the total mass of the system, while the centred wave develops into an ever-thinner sheet of fluid with ever-increasing velocity, and thus with ever-increasing Froude number. This situation is even more physically unrealistic than that obtaining in rectilinear flow.

In view of the physically unrealistic nature of the transient corresponding to a St. Venant front, we have also computed the transient for the Jeffreys-Vedernikov front corresponding to a de Chézy resistance law, so that $u_{\text{front}} = 2\sqrt{g h_{\text{front}}}$. This flow is then nearly-horizontal "almost everywhere". The solution is again unique for all u_0 and $h_0 = h(u_0)$ and is accordingly presented in dimensionless form. We remark that, in this solution, a region of steady state is again formed adjacent to the boundary, and, as this extends in time, so the Froude number increases at its junction with the remainder of the solution. As this remainder contains a front characterized by a Froude number, a hydraulic jump is formed in the vicinity of the junction.

As radial flows are the simplest and most ideal of all two-dimensional flows, we may anticipate that their transients will present the particular properties and difficulties of two-dimensional flows in an especially clear way. The present solutions then suggest that the assumptions underlying the various front conditions should be investigated most carefully, as they provide solutions that differ essentially. These differences will clearly subsist even when resistance terms are introduced together with the St. Venant front and will, of course, be most pronounced in stratified fluids, where fronts are inevitable.

The solutions further indicate that fronts characterized by Froude number will increase that tendency to form hydraulic jumps that is always present in non-diverging quasi-linear flows. In numerical studies this appears to present a major difficulty, since no genuine system of conservation laws can be written for two-dimensional flows. The usual numerical approach, through "dissipative" difference schemes (e.g. Richtmyer and Morton, 1967) is then no longer applicable.

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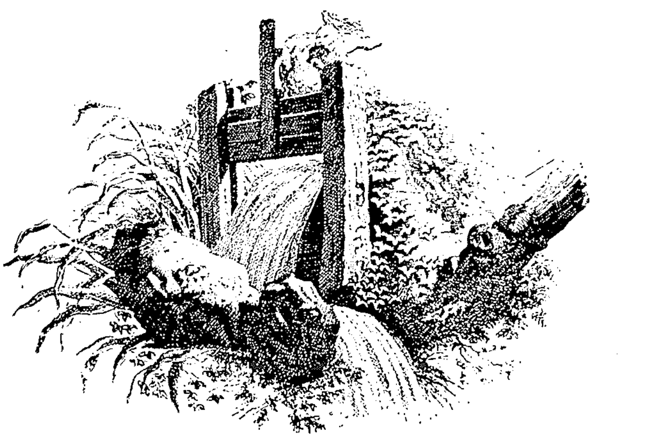

Gravure du XVIIIe siècle