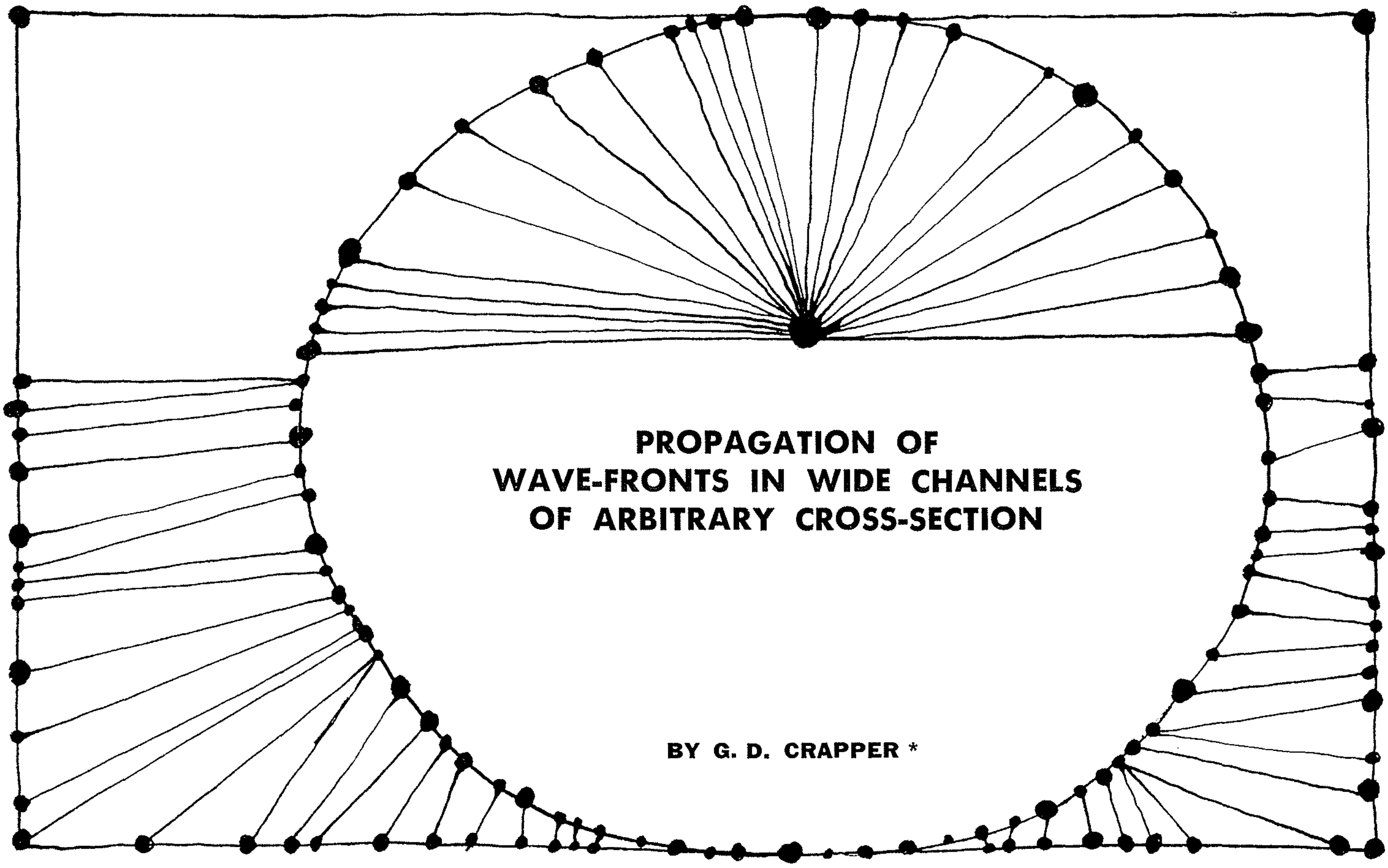

PROPAGATION OF WAVE-FRONTS IN WIDE CHANNELS OF ARBITRARY CROSS-SECTION

BY G. D. CRAPPER *

This paper applies to channel hydraulics a recent paper on partial differential equation theory. Equations are presented in a general form for any shape of channel, and although the derivation of these equations is very complicated, and is therefore omitted, their application is relatively simple. They are applied here to a channel of triangular cross-section and show that initially the disturbance moves away to the sides of the channel.

I. — Introduction

A recent paper, Varley and Cumberbatch [1], has provided a simple method for predicting the movement of a wave-front and a measure of its strength in a system governed by hyperbolic partial differential equations. An application of this method with interest to civil engineers is the propagation of a change in the flow of water in wide, shallow channels with non-rectangular cross-section. The equations are presented here in a manner which allows them to be easily applied to any configuration, including variation of depth along as well as across the channel. The only drawback of the method is that it gives no information on the flow behind the front. However in a necessarily complicated problem, with two space variables and time involved, it provides a relatively simple method of obtaining some information about the flow.

The particular case discussed is that in which the channel cross-section is triangular, and some rather unusual results are predicted. For simplicity a front is considered to be a discontinuity of the slope of the free surface, although a discontinuity of curvature or of a higher derivative could equally well be taken, and the front is shown to split along the centre-line of the channel with all the initial disturbance moving away to the sides of the channel. The variation of strength of the discontinuity is calculated and the wave-front breaks to form a bore where this becomes infinite. These particular results are confirmed by a solution for a channel which has a hyperbola as cross-section, the eccentricity of the hyperbola being a variable so that it can be allowed to tend to a triangle.

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II. — Equations

Axes of x and y are taken in a horizontal plane. The equations of motion and continuity may then be written:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial}{\partial x}(h - h_0) + \frac{gu}{F^2 h}\sqrt{u^2+v^2} = 0\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial}{\partial y}(h - h_0) + \frac{gv}{F^2 h}\sqrt{u^2+v^2} = 0\\
&\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\frac{\partial u}{\partial x} + h\frac{\partial v}{\partial y} = 0
\end{aligned}\right\} \quad (1)$$

where u, v are the x, y velocity components, h is the total depth, h_0 the original undisturbed depth, g the gravitational acceleration and F the Chézy friction coefficient.

Following Varley and Cumberbatch we define a wave front as a line along which velocity and depth derivatives may be discontinuous, i.e. a wave front is a characteristic surface of the equations. Characteristic surfaces are determined by the vanishing of the characteristic determinant of the equations (1) which yields the equation

$$(un_1 + vn_2 - \mathrm{V})\,[un_1 + vn_2 - \mathrm{V} + \sqrt{gh\,(n_1^2 + n_2^2)}]\,[un_1 + vn_2 - \mathrm{V} - \sqrt{gh\,(n_1^2 + n_2^2)}] = 0 \qquad (2)$$

where (n_1, n_2) is the unit normal vector to the front and V the speed of the front along this normal. As (2) gives three real distinct values for V the equations are totally hyperbolic. (Alternative derivations of this determinant may be found in Abbott [2] or Daubert and Graffe [3].)

From (2) we select on physical grounds:

$$\mathrm{V} = un_1 + vn_2 + \sqrt{gh\,(n_1^2 + n_2^2)} \qquad (3)$$

which (since $n_1^2 + n_2^2 = 1$) represents propagation with speed $c = \sqrt{gh}$ relative to the flow. From (3) bicharacteristic equations are derived which determine how a particular characteristic surface moves. Varley and Cumberbatch show that if $t\,(x, y)$ is the time at which such a surface arrives at point (x, y) then $\partial t/\partial x = n_1/\mathrm{V}$ and $\partial t/\partial y = n_2/\mathrm{V}$. Thus (3) is a first order equation for t and can be reduced to ordinary differential equations by standard methods. The resulting equations for the bicharacteristics are:

$$\frac{dx}{dt} = u + c\cos\beta \qquad \frac{dy}{dt} = v + c\sin\beta$$

$$\frac{d\beta}{dt} = \cos\beta\left(\sin\beta\,\frac{\partial u}{\partial x} - \cos\beta\,\frac{\partial u}{\partial y}\right) + \sin\beta\left(\sin\beta\,\frac{\partial v}{\partial x} - \cos\beta\,\frac{\partial v}{\partial y}\right) + \sin\beta\,\frac{\partial c}{\partial x} - \cos\beta\,\frac{\partial c}{\partial y} \qquad (4)$$

where $c = \sqrt{gh}$, $n_1 = \cos\beta$ and $n_2 = \sin\beta$. Note that these equations are not affected by friction. It can also be shown that the right hand sides of (4) contain derivatives only in combinations which are continuous at the wave-front. Thus in solving them the known values ahead of the front may be used. In particular $c = \sqrt{gh_0}$, where, of course, h_0 is the known undisturbed depth, and:

$$\partial c/\partial x = (g\partial h_0/\partial x)/2\,c$$

Varley and Cumberbatch go on to derive an equation for the variation along a bicharacteristic of the strength s of the front, defined as the jump in surface slope at the front, the slope being zero ahead of the front. The derivation is very complicated and only the result is given here; it is:

$$\frac{d\alpha}{dt} + \mathrm{D}\alpha - \frac{3\,c}{2\,\mathrm{V}}\,\alpha^2 = 0 \qquad (5)$$

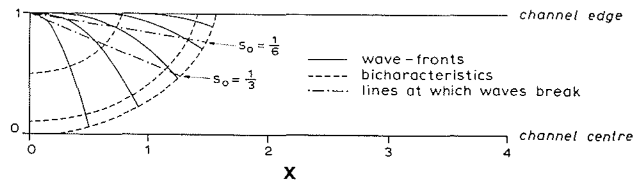
where:

$$\alpha = \mathrm{V}s/c^2 \qquad \mathrm{V} = u\cos\beta + v\sin\beta + c \qquad (6)$$

$$\mathrm{D} = \frac{1}{2\,c}\left\{ u\,\frac{\partial c}{\partial x} + v\,\frac{\partial c}{\partial y} + c\cos^2\beta\,\frac{\partial u}{\partial x} + c\sin^2\beta\,\frac{\partial v}{\partial y} + c\sin\beta\cos\beta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 5\,c\left(\cos\beta\,\frac{\partial c}{\partial x} + \sin\beta\,\frac{\partial c}{\partial y}\right) - \frac{c^2}{\mathrm{R}}\right\} + \frac{1}{\mathrm{F}^2 c\sqrt{u^2 + v^2}}\left\{ u^2 + v^2 + (u\cos\beta + v\sin\beta)^2 - c\,(u^2 + v^2)\,(u\cos\beta + v\sin\beta)\right\} \qquad (7)$$

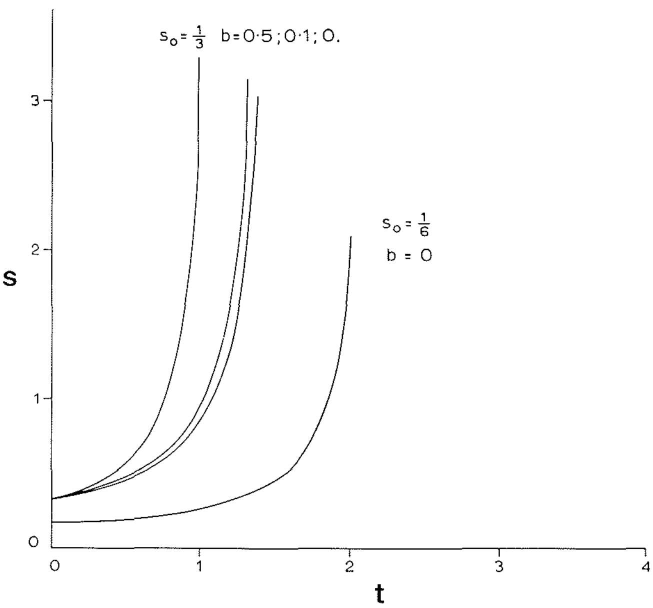
and:

$$\frac{1}{\mathrm{R}} = \sin\beta\,\frac{\partial\beta}{\partial x} - \cos\beta\,\frac{\partial\beta}{\partial y} \qquad (8)$$



1/ Wave fronts, bicharacteristics and breaking lines in a triangular channel, with two values of the initial steepness s_0. The wave fronts shown are for $t = 0.5$; 1; 1.5; 2; 2.5 and the bicharacteristics begin at $(0, b)$ for $b = 0$; 0.1; 0.5. In all figures the unit of x, y is B, the unit of t is $\mathrm{B}/\sqrt{g\mathrm{H}}$ and the unit of s is H/B.

Fronts d'onde, bi-caractéristiques et lignes de déferlement, dans un canal de section triangulaire, en fonction de deux valeurs différentes de la cambrure initiale s_0. *Les fronts d'onde représentés correspondent à* $t = 0{,}5$; 1; 1,5; 2; 2,5, *et les bi-caractéristiques partent de* (0, b) *avec* b = 0; 0,1; 0,5. *Dans toutes les figures, B est l'unité de* x, y, $B/\sqrt{gH}$ *celle de* t, *et H/B celle de* s.



2/ Steepness s as a function of t on the bicharacteristics shown in figure 1.

Cambrure s *en fonction de* t, *correspondant aux bi-caractéristiques de la figure 1.*

R is the radius of curvature of the wave-front. This time friction appears in the equation. Again values ahead of the front can be used, and the coefficient D is greatly simplified if u and v are either zero or constant in that region. In deriving (5) we have assumed that time derivatives of u, v and h are zero ahead of the front, i.e. the front is moving into a region of steady flow.

III. — Solution for a channel of triangular cross-section

We consider only the case in which the flow ahead of the front is (U, 0), U constant, and the undisturbed channel depth is independent of x, so that $\partial c/\partial x = 0$

Equation (4) for the bicharacteristics can be solved simply if we take β rather than t as a parameter on a bicharacteristic. If the initial wave front at $t = 0$ is $x = 0$, we have:

$$c = \mathrm{C} \cos \beta \tag{9}$$

$$x = -\int_0^{\beta} \frac{(u + \mathrm{C} \cos^2 \beta)}{\cos \beta \, (\partial c/\partial y)} \, d\beta \tag{10}$$

and:

$$t = -\int_0^{\beta} \frac{d\beta}{\cos \beta \, (\partial c/\partial y)} \tag{11}$$

where $\mathrm{C} = c_0(b)$ for a bicharacteristic beginning at (0,b). In (10) and (11) $\partial c/\partial y$ is expressed as a function of β from (9), and (9) also determines y as a function of β. These equations do not hold in regions of level bottom where $\partial c/\partial y = 0$, but there integration of (4) is trivial. For a fixed C these equations give the bicharacteristics. However if β is eliminated to give x and y as functions of t then the lines given when $t =$ constant are the positions of the wave front at that time.

For a triangular channel of depth H and width 2 B:

$$c = \sqrt{g\mathrm{H}\left(1 - \frac{y}{\mathrm{B}}\right)}, \quad 0 < y < \mathrm{B} \tag{12}$$

(the negative half follows by symmetry) and (9), (10), (11) give:

$$\begin{aligned} t &= 2\,\mathrm{C}\beta \\ x &= \mathrm{U}t + \frac{1}{2}\mathrm{C}^2 \sin\frac{t}{\mathrm{C}} + \frac{1}{2}\mathrm{C}t \\ y &= 1 - \frac{1}{2}\mathrm{C}^2 - \frac{1}{2}\mathrm{C}^2 \cos\frac{t}{\mathrm{C}} \end{aligned} \tag{13}$$

When $\mathrm{U} = 0$, which is the only case for which results are presented, the bicharacteristics C = constant are cycloids and the wave fronts are orthogonal to them, as shown in figure 1. The interesting feature of this result is that the wave front splits at the channel centre leaving, for $t > 0$, a gap which looks as though it should be filled with some sort of 'expansion fan' of bicharacteristics. If we consider the flow immediately behind the front we can show that it is normal to the front and therefore fluid initially moves out behind the front, apparently leaving no flow in the channel centre where there is no front; of course the theory gives only conditions actually at the front, and presumably some flow moves in to the centre from the limiting bicharacteristic.

The variation of strength s along a particular bicharacteristic has to be calculated numerically from (5), first re-writing the curvature as:

$$\frac{1}{\mathrm{R}} = \frac{\partial\beta}{\partial \mathrm{C}} \left\{ \left(\frac{\partial x}{\partial \mathrm{C}}\right)^2 + \left(\frac{\partial y}{\partial \mathrm{C}}\right)^2 \right\}^{-1/2} \tag{14}$$

where $\partial/\partial\mathrm{C}$ indicates that t is held constant. Results obtained by using a standard library programme for the KDF 9 computer are shown in figure 2 for particular initial values s_0 of s (s_0 taken constant across the channel). Since we consider $\mathrm{U} = 0$ only, the friction term again drops out. The front 'breaks' when α and hence s becomes infinite, and points at which this happens can be shown analytically to lie on a straight line. These lines are shown in figure 1. If a negative initial value for s had been used, the magnitude of s would simply have decreased along the bicharacteristics.

IV. — Solution for a channel of hyperbolic cross-section

To give further information on the splitting of the front we now consider a channel which has no slope discontinuity. The hyperbolic cross section:

$$h = \frac{1}{2}\,\mathrm{H}\left[1 + \varepsilon - \left\{ (\varepsilon - 1)^2 + 4\,\varepsilon\,\frac{y^2}{\mathrm{B}^2} \right\}^{-1/2} \right]$$

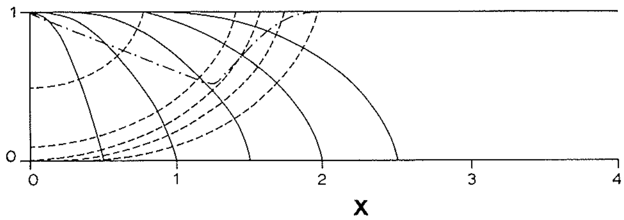
$$\varepsilon \geqslant 1 \tag{15}$$

is suitable, because as $\varepsilon \rightarrow 1$ the shape tends to the triangular section. This time the integrals in (10) and (11) are not simple so the whole problem has to be solved numerically. For numerical work the initial value of y, b, is more convenient than C as a parameter, and in (5) we then have:

$$\frac{1}{\mathrm{R}} = -\frac{\partial\beta}{\partial b} \left\{ \left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2 \right\}^{-1/2} \tag{16}$$

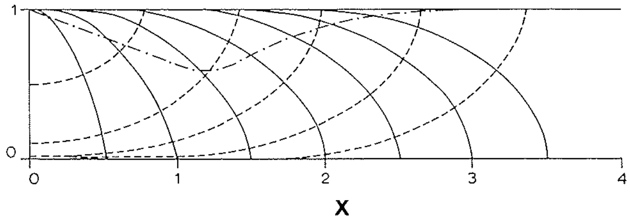
—note the minus, compared with (14), because b and C increase in opposite directions. To obtain the derivatives in (16) numerically they have to be considered as new variables and equations for their variation along a bicharacteristic derived by differentiating (4) with respect to b. With no initial flow and c a function of y only this gives:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial x}{\partial b}\right) &= \frac{dc}{dy}\frac{\partial y}{\partial b}\cos\beta - c\sin\beta\,\frac{\partial\beta}{\partial b} \\ \frac{d}{dt}\left(\frac{\partial y}{\partial b}\right) &= \frac{dc}{dy}\frac{\partial y}{\partial b}\sin\beta + c\cos\beta\,\frac{\partial\beta}{\partial b} \\ \frac{d}{dt}\left(\frac{\partial\beta}{\partial b}\right) &= -\frac{d^2c}{dy^2}\frac{\partial y}{\partial b}\cos\beta + \frac{dc}{dy}\sin\beta\,\frac{\partial\beta}{\partial b} \end{aligned} \tag{17}$$



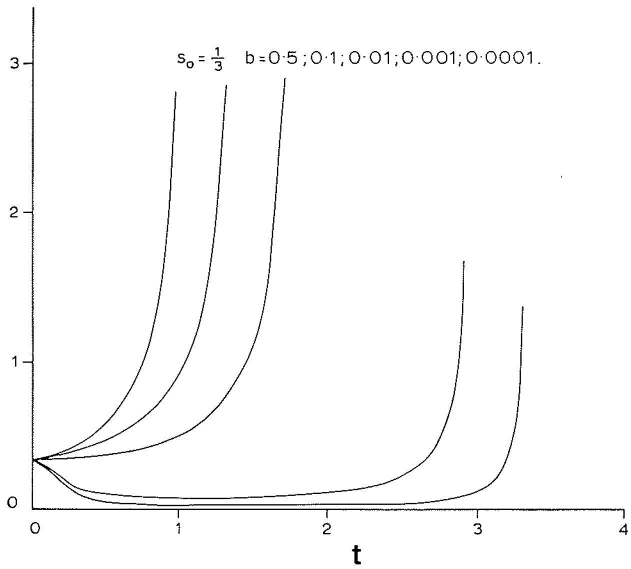
3/ Wave fronts, bicharacteristics and breaking line in a hyperbolic channel with $\varepsilon = 1.01$ and initial steepness $s_0 = 1/3$. The wave fronts shown are for $t = 0.5$; 1; 1.5; 2; 2.5 and the bicharacteristics for $b = 0.0001$; 0.001; 0.01; 0.1; 0.5.

Fronts d'onde, bi-caractéristiques et ligne de déferlement, dans un canal hyperbolique avec ε = 1,01, et une cambrure initiale s₀ = 1/3. Les fronts d'onde représentés correspondent à t = 0,5; 1; 1,5; 2; 2,5, et les bi-caractéristiques à b = 0,0001; 0,001; 0,01; 0,1; 0,5.



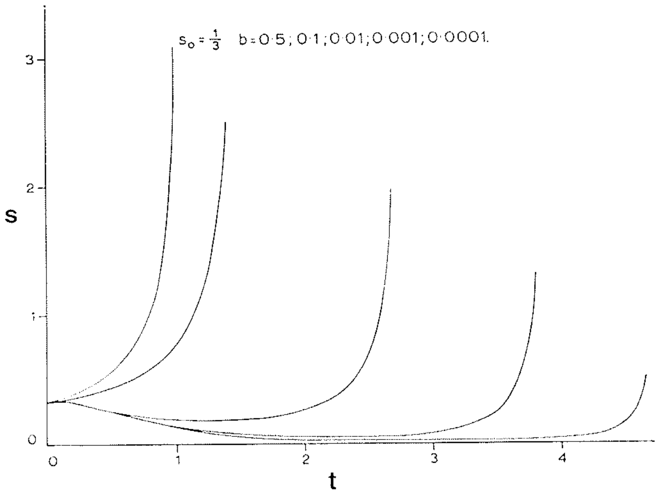
5/ Wave fronts, bicharacteristics and breaking line in a hyperbolic channel with $\varepsilon = 1.1$ and initial steepness $s_0 = 1/3$. The wave fronts shown are for $t = 0.5$; 1; 1.5; 2; 2.5; 3; 3.5 and the bicharacteristics for $b = 0.0001$; 0.001; 0.01; 0.1; 0.5.

Fronts d'onde, bi-caractéristiques et ligne de déferlement dans un canal hyperbolique, avec $\varepsilon = 1,1$ et une cambrure initiale $s_0 = 1/3$. Les fronts d'onde représentés correspondent à t = 0,5; 1; 1,5; 2; 2,5; 3; 3,5, et les bi-caractéristiques à b = 0,0001; 0,001; 0,01; 0,5.



4/ Steepness s as a function of t on the bicharacteristics shown in figure 3. Note the small values of s near the channel centre.

Cambrure s en fonction de t, correspondant aux bi-caractéristiques représentées sur la figure 3. On notera les faibles valeurs de s au voisinage de l'axe du canal.



6/ Steepness s as a function of t on the bicharacteristics shown in figure 5. Again the values are very small near the channel centre.

Cambrure s en fonction de t, correspondant aux bi-caractéristiques représentées sur la figure 5. Dans ce cas également, les valeurs restent très faibles au voisinage de l'axe du canal.

Equations (4), (17) and (5) can then be solved using the same library programme as before. The results are plotted for two values of ε in figures 3 to 6. In each case the channel is almost indistinguishable from a triangular section except for a very small rounding off of the discontinuity at $y = 0$. The results of the triangular case are essentially confirmed. Although the front now goes right across the channel, it will be noted that the strength s of the front decreases rapidly in the region which corresponds to the gap in the front in the earlier case, confirming that in that case there is no initial flow in this region.

These results suggest that channels of triangular cross-section would be very effective in removing energy from disturbances propagating up them. Experiments on such a channel are being carried out at Leeds University by Mr. P. B. Hird and will be reported in due course in the appropriate journal, toghether with more detailed calculations of the flow behind the front. However early results confirm the above work.

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