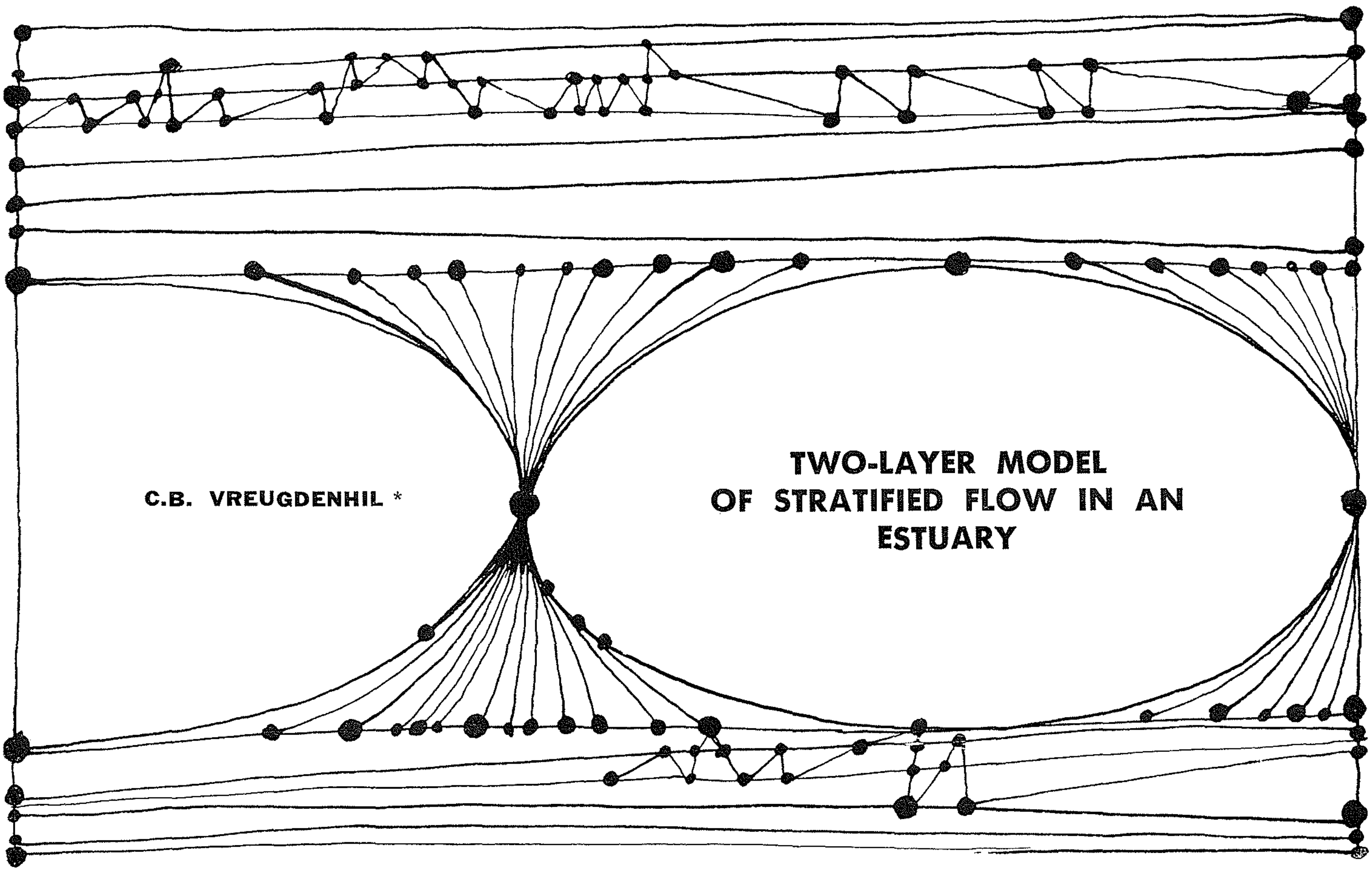

TWO-LAYER MODEL OF STRATIFIED FLOW IN AN ESTUARY

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I. — Introduction

As the available knowledge of processes in turbulent stratified flow is rather limited, computational methods must be more or less schematic. This does not exclude useful and even relatively detailed results. In case tidal variations are important, two approximate methods can be mentioned. For well-mixed estuaries a one-dimensional model is available, as described by Stigter and Siemons [10]. For fully stratified flow, Schijf and Schönfeld [9] formulated a two-layer mathematical model. This model has been applied by Boulot c.s. [1, 2] and by the present author [12]. Both models have a wider applicability than indicated by the extremes of the schematization. This has been demonstrated for the one-dimensional model in [10]. It is shown for the two-layer model in the present paper.

Essentially a two-layer schematization is possible for any degree of stratification. Depending on the definition of the "interface" the following unknown quantities are required: a frictional coefficient at the interface and the quantities of water and salt exchanged through the interface. The success of the computations depends on the accuracy with which one specifies these quantities. In this paper the exchange of salt and water is neglected. More elaborate models may be required for better accuracy.

The computations are verified by means of prototype data from the Rotterdam Waterway [7]. At the time of the measurement (1956) this constituted an almost uniform channel. From the measurements it can be concluded that the estuary belongs to the partly mixed class. This is borne out by the values of some parameters from literature [3, 4, 8], which indicate the same classification (table 1). The notation is explained in section 8.

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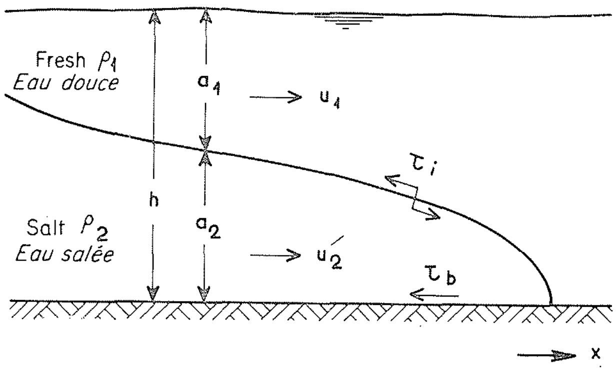
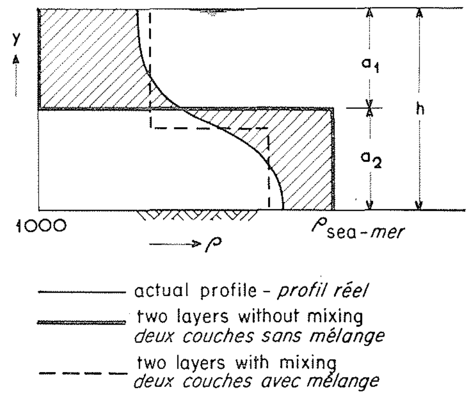
Classification of the Rotterdam Waterway (1956) **Table 1**

AUTHOR	PARAMETER	VALUE FOR ROTTERDAM WATERWAY '56	BOUNDS FOR CASE: Fully stratified	BOUNDS FOR CASE: Well mixed
Schultz & Simmons [8]	flow ratio = tidal prism/river discharge during flood tide	~ 0.2	< 0.1	> 1
Harleman & Ippen [4]	estuary number: $P_t u_0^2 (Q_f T g h)^{-1}$	~ 0.015	?	> 0.15
Hansen & Rattray [3]	$P = u_f / u_t$ $F_m = u_f / (\varepsilon g h)^{1/2}$	~ 0.16 ~ 0.1	partly mixed according to Figure 2 of [3].	



1/ Definitions.
Définitions.



2/ Schematization.
Schématisation.

II. — Mathematical description

The differential equations for the two-layer model without exchange can be obtained by integration in vertical direction of the local equations of continuity and motion. Neglecting the transport of salt and water across the interface, the equations given by Schijf and Schönfeld are obtained:

$$\frac{\partial a_1}{\partial t} + \frac{\partial}{\partial x}(a_1 u_1) = 0 \tag{1}$$

$$\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(a_2 u_2) = 0 \tag{2}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + g \frac{\partial h}{\partial x} + \frac{\tau_i}{\rho_1 a_1} = 0 \tag{3}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + (1 - \varepsilon)\, g \frac{\partial h}{\partial x} + \varepsilon g \frac{\partial a_2}{\partial x} + \frac{\tau_b - \tau_i}{\rho_2 a_2} = 0 \tag{4}$$

In these equations lateral variations are neglected.

This process of integration over the two layers is possible for any definition of the interface and for any degree of stratification. Of course the approximations involved in (3) and (4) do depend on the stratification. The most important ones are made in the terms, representing the slopes of the surface and the interface. This is caused by the assumption of a hydro-static pressure distribution, corresponding to the schematized density-profile. In case the mixing between the layers is taken into account, this approximation is less serious (cf. fig. 2).

Due to the assumption of no exchange, the densities in the layers are those of fresh water ρ_1 and sea water ρ_2. The mean density in a vertical is $(a_1\rho_1 + a_2\rho_2)/h$. In order to compare the computational results to the prototype values, the layer-thickness for the prototype should be defined such that at least the mean densities are comparable:

$$a_1\rho_1 + a_2\rho_2 = h\bar{\rho}$$

or:

$$a_2 = h\, \frac{\bar{\rho} - \rho_1}{\rho_2 - \rho_1} \text{ (prototype)} \tag{5}$$

This is illustrated in Figure 2. In the more general case the layer densities ρ_1 and ρ_2 may vary too. A much closer approximation of the density-profile is possible then.

The frictional stresses τ_i and τ_b in (3) and (4) are defined by:

$$\tau_i = \rho_2 k_i (u_1 - u_2)|u_1 - u_2| \tag{6}$$

$$\tau_b = \rho_2 k_b u_2 |u_2| \tag{7}$$

The frictional coefficients k_i and k_b are empirical functions of the global and possibly local conditions in the estuary.

III. — Schematization

The system of differential equations (1)...(4) results in two types of waves, characterized as surface and internal waves. With some approximation their velocities of propagation are respectively:

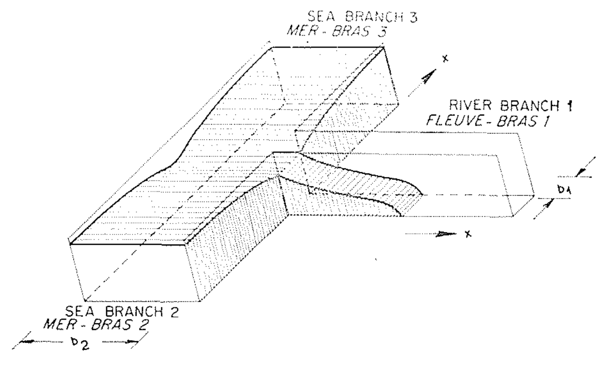
$$c_s = \frac{a_1 u_1 + a_2 u_2}{h} \pm \sqrt{gh} \tag{8}$$

$$c_i = \frac{a_1 u_2 + a_2 u_1}{h} \pm \sqrt{\varepsilon g a_2 \frac{a_1}{h} (1 - F_i^2)} \tag{9}$$

where $F_i = (u_1 - u_2)(\varepsilon g h)^{-1/2}$ is a Froude-number with respect to the internal waves. As ε usually is less than 0.03 and F_i may be appreciable, the order of magnitude of c_s may be twenty times c_i. This means that the surface waves are considerably longer than internal waves with the same period.

For a numerical computation, using a finite-difference technique, it is necessary that the ratio $c\Delta t/\Delta x$ does not differ too much from unity for the wave considered, in order to have some accuracy. On the other hand, an explicit difference-method requires step-widths such that $c\Delta t/\Delta x < 1$ for any wave [11]. Evidently for the present system these conditions are contradictory due to the difference in magnitude between c_i and c_s. An implicit difference-method, which is not subject to the latter condition, is unattractive for the non-linear system (1)...(4). Therefore to set up a reasonable computation for the internal waves it is desirable to eliminate the surface waves. This can be done using the fact that the surface waves are much longer than the internal waves. As an approximation the surface is assumed to be horizontal:

$$a_1 + a_2 = h(x, t) = h(0, t) \tag{10}$$



3/ System in schematized-sea method.
Méthode avec schématisation de la zone marine.

By continuity, from the sum of equations (1) and (2), the net discharge q becomes:

$$a_1 u_1 + a_2 u_2 = q(x, t) = q_L + (L - x) \frac{dh}{dt} \qquad (11)$$

It is not justified simply to neglect the surface slope in equation (4), as it may have a considerable influence. However, it is possible to eliminate it by subtracting $(1 - \varepsilon)$ times equation (3) from equation (4). Neglecting ε with respect to unity at some unimportant points, one finds:

$$\frac{\partial}{\partial t}\left(u_2 - u_1\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\, u_2^2 - \frac{1}{2}\, u_1^2 + \varepsilon g a_2\right) + $$
$$+ \frac{\tau_b}{\rho_2 a_2} - \frac{\tau_i}{\rho_2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) = 0 \qquad (12)$$

Now h and q are known from (10), (11) and two boundary conditions (cf. section 4). Then a_1 and u_1 can be expressed in terms of a_2 and u_2 using the left hand sides of (10) and (11). The remaining unknowns a_2 and u_2 can be solved from equations (2) and (12). This is a second order system, which shows the velocities c_i from equation (9) exactly. In this approximation, which can be justified systematically, the main tidal influence is retained, but the velocity of propagation of the surface waves is assumed to be infinite.

The system (2), (12) is solved using the Lax-Wendroff finite-difference method [5].

IV. — Boundary conditions

For the surface waves two boundary conditions are required: the fresh-water discharge q_f and the water-level $h(0, t)$ at the river-mouth. The latter is introduced into (10). The former can be taken into account in two ways. The simplest method is to identify q_L in (11) with q_f and take L so large that the horizontal tide at the river-mouth is approximately correct. A more accurate method is to compute q_L as a function of time using a tidal computation for homogeneous water. The location $x = L$ then appears to be rather arbitrary.

The upstream boundary condition for the lower layer simply reads that the layer-thickness vanishes. It is not necessary to treat the front of the salt wedge separately. Upstream of this front, the equations (2) and (12) are satisfied by:

$$a_2 = 0 \quad \text{and} \quad u_2 = u_1 \qquad (13)$$

provided the bottom-friction vanishes too *. The Lax-Wendroff method takes care of possible steep fronts, so it can be applied throughout the region.

The conditions for the internal flow at the river-mouth are more complicated. Two methods are devised and the possible situations are summarized in table 2, depending on the behaviour of the velocities c_{i1} and c_{i2} of internal waves.

Boundary conditions at the river mouth **Table 2**

	Condition	Character	Boundary conditions: Critical flow method	Boundary conditions: Schematized-sea method
1	$c_{i1} < c_{i2} < 0$	supercritical outflow	none	none
2	$c_{i1} < 0 < c_{i2}$	subcritical flow	critical flow: $c_{i2} = 0$	continuity + outflow condition
3	$0 < c_{i1} < c_{i2}$	supercritical inflow	doubly critical flow: $c_{i1} = c_{i2} = 0$	doubly critical flow: $c_{i1} = c_{i2} = 0$

As to the critical flow method, in a steady situation critical flow at the river mouth can be made plausible. In presence of the tide, however, the flow probably cannot adjust itself to critical flow at each moment. This is due to the large time-scale for the internal flow (noted also by Boulot and Daubert [2]). Yet critical flow could be applied as a working hypothesis. In the case with mixing between the layers, this condition is not sufficient, as in addition the density of the inflowing water has to be specified.

The schematized-sea method corresponds to the conditions near the Rotterdam Waterway, where the tidal flow is roughly parallel to the coast. The sea is schematized as a one-dimensional channel (fig. 3). At the river-mouth continuity is taken into account, together with an outflow condition. A similar method is applicable to the case with mixing.

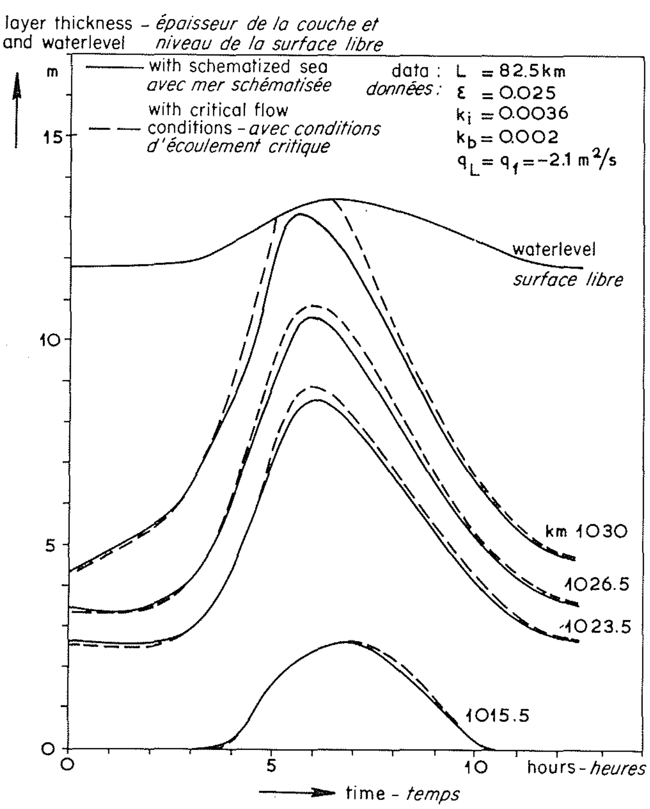
The situation of supercritical inflow is treated by analogy with homogeneous supercritical flow. In the steady state the condition of doubly critical flow corresponds to a maximum of the return-current [6]. The condition $c_{i1} = c_{i2} = 0$ results in :

$$a_2(0, t) = \frac{1}{2}\, h(0, t) + \frac{1}{2}\, q(0, t)\, [\varepsilon g h(0, t)]^{-1/2}$$
$$u_2(0, t) = q(0, t)/2\, h(0, t) + \frac{1}{2}\, [\varepsilon g h(0, t)]^{+1/2} \qquad (14)$$

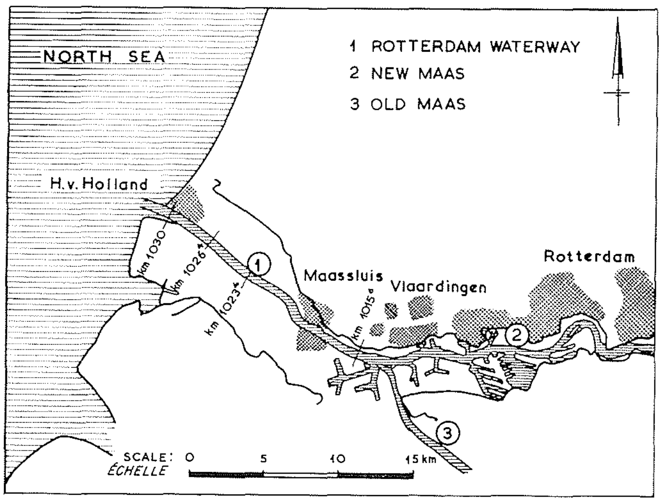
The initial condition is steady flow, corresponding to the river discharge. Its influence is assumed to vanish after a sufficient interval of time.

A comparison between the critical-flow method and the schematized-sea method is made in Figure 4. The results of computations for the Rotterdam Waterway (section 5)

* Actually the interfacial friction takes over the role of the bottom friction.



4/ Comparison of methods for boundary conditions.
Comparaison des méthodes pour les conditions aux limites.



5/ Rotterdam waterway.
Le complexe de Rotterdam.

are shown for the two methods, using the same data. The vertical tide was measured at 15-6-1966. The length L was chosen such that the maximum flood discharge is correct. It is seen that there is very little difference in the computed layer-thickness. Therefore in the present case the simpler critical-flow method is applied.

V. — Application to the Rotterdam Waterway

In 1956 the Rotterdam Waterway in its most seaward 18 km was a relatively uniform channel (fig. 5). Computations were done for the measurements at 22-6-1956 using the data as given in table 3.

The data for cases A and B are identical, except of course q_L. Results are shown in Figures 6 to 9.

The effect of applying a tidal computation to obtain q_L is shown in Figure 6. The peak-values for case A are somewhat too small. This could be improved by a more careful tidal computation. Partially the effect is due to the assumption of a horizontal water-level. In case B the length L has been adjusted such that the flood and ebb volumes agree with case A. This results in a not very realistic shape of the horizontal tidal curve. The effect of this can be seen in the subsequent figures.

The interfacial frictional coefficient k_i has been adjusted to obtain a good correspondence between measurements and computations. This has been done mainly with respect to the layer-thickness in Figure 7. The curve for km 1,015.5 has not been given too much weight as this section is rather close to the branching point with the Old Maas (fig. 5). Apart from this a quite reasonable agreement with the measurements is seen. Near high water and just before low water case B shows some deviations due to the peaked curve for the discharge q (fig. 6).

The phase-shift of 2 h for case B, shown in Figure 7 has been applied in all figures to account for the phase-shift between horizontal and vertical tide. This phase-shift is reproduced faithfully in case A.

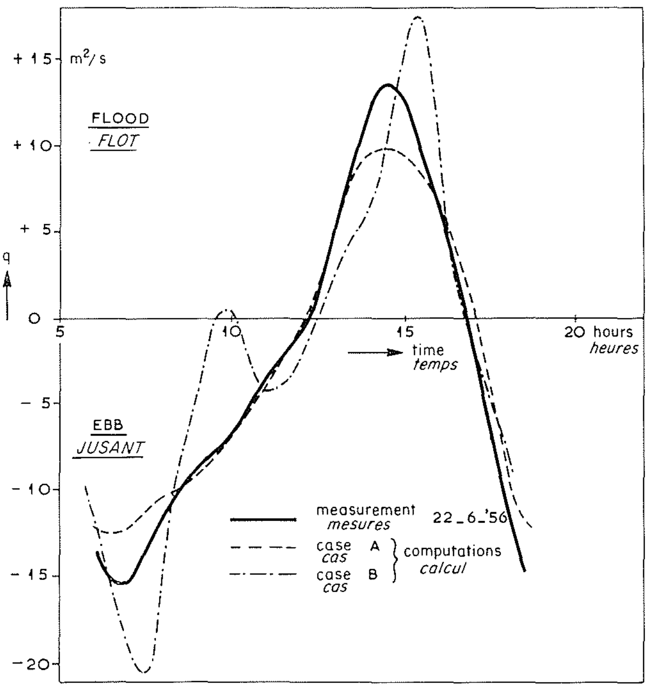
The results of Figure 7 are presented in a different form in Figure 8. The shape of the salt wedge is seen to be reasonable, except at 7 h. Case A tends to be slightly better (e.g. at 17 h) but the difference between the two methods is small.

In Figure 9 the horizontal velocity in the lower layer at km 1,030 is shown. It is reproduced satisfactorily in case A except at low water. Case B is worse as could be expected from Figure 6. Yet the order of magnitude is right.

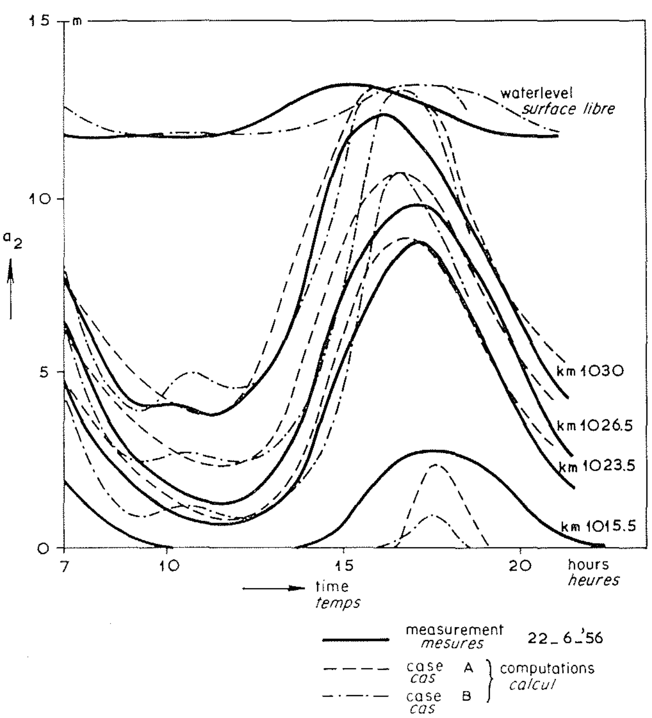
In all figures it should be realised that the measurements include some inaccuracies, generally in the order of 10 %.

Data for computations **Table 3**

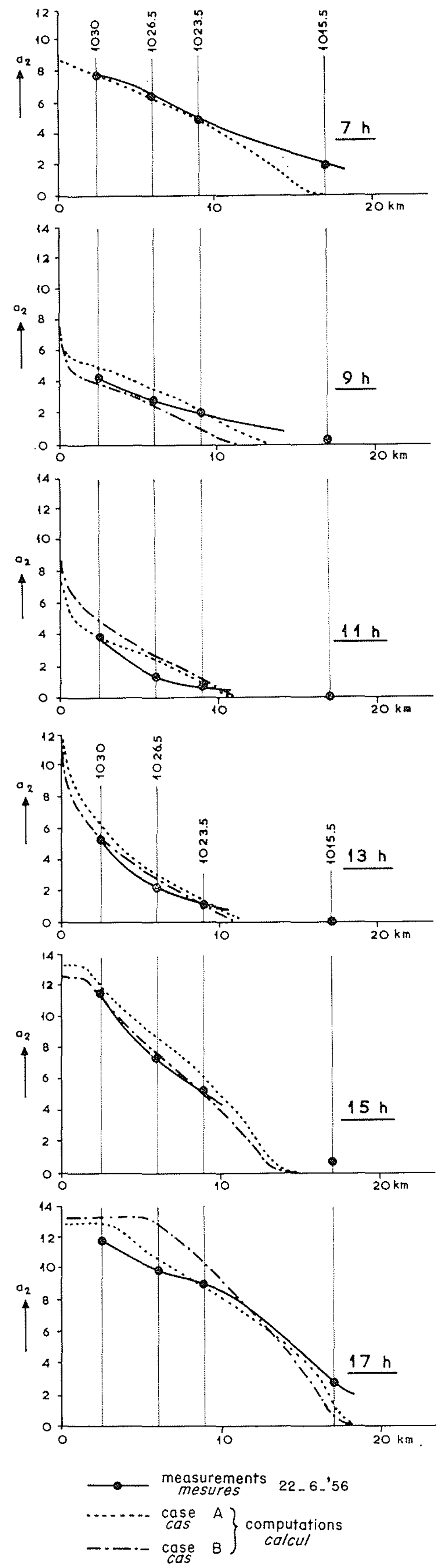
Dimensions of cross-section	12.5 × 480 m	Boundary condition at river-mouth	critical flow method
Vertical tide from measurements (22 - 6 - '56).		Step widths	Δx 500 m Δt 180 s
Density of sea-water ρ_2	1 025 kg/m³	*Case A* (with tidal computation): L = 27 km, q_L computed	
Density of river-water ρ_1	1 000 kg/m³	*Case B* (without tidal computation): L = 113 km, q_L = — 2.1 m²/s	
River-discharge per unit width	2.1 m²/s		
Chézy-coefficient C	60 m$^{1/2}$/s		
Coefficient of bottom friction k_b	0.0028		
Coefficient of interfacial friction k_i	0.0052		



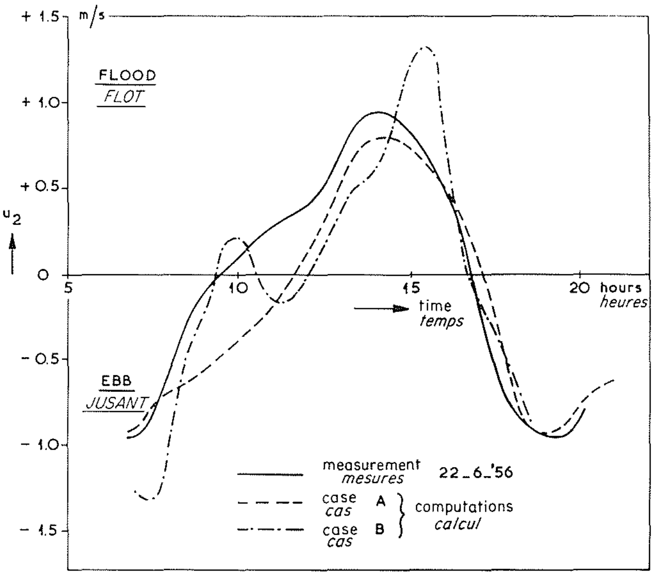
6/ Discharge at km 1030.
Débit au km 1030.



7/ Layer-thickness.
Epaisseur de la couche.



8/ Shape of the salt wedge.
Forme du coin salé.



9/ Velocity in the lower layer at km 1 030.
Vitesse dans la couche inférieure au km 1 030.

VI. — Discussion

By comparison of cases A and B in the preceding section, it is concluded that the *horizontal* tide is the main factor influencing the salt wedge. The vertical tide being 2 h late in case B is noticed hardly. Therefore a good reproduction of the horizontal tide $q(x, t)$ is essential. The accuracy, obtained by assuming a horizontal water-level is satisfactory, provided the boundary value $q_L(t)$ is computed properly as in case A. For a simpler shape of the vertical tide the method of case B may be sufficient.

A second important factor is the seaward boundary condition for the internal flow. Although differences in the treatment may not be important (fig. 4), the essential processes of supercritical in and outflow should be included.

The evaluation of the correspondence between measurements and computations depends on the required accuracy. As Figures 6 to 9 show a reasonable reproduction of the layer-thickness (mean density, eq. 5) and velocities, the method is considered to be useful for many practical purposes, in which these overall quantities are sufficient. More-over it is possible that the velocity-profiles in the vertical can be estimated if the mean values in upper and lower layer are known. Essentially the value of the two-layer model is determined by the empirical parameter k_i. At present this parameter cannot be predicted. However by hindcasting several known situations, its behaviour can be estimated depending on the overall conditions of the estuary (river discharge, tidal amplitude, etc.).

In principle a better accuracy is possible if the exchange of salt and water between the layers is taken into account. This would also give a better impression of the density-profile. The price for this accuracy however is the necessity of predicting more empirical parameters. Therefore the two-layer model without mixing probably is the most attractive one in most cases.

VII. — Notations

a_1 : upper-layer thickness;
a_2 : lower-layer thickness;
c_i : velocity of propagation for internal waves;
c_s : velocity of propagation for surface waves;
C : Chézy-coefficient;
F_i , F_m : internal Froude number;
g : acceleration due to gravity;
h : waterdepth;
k_b : coefficient of bottom-friction;
k_i : coefficient of interfacial friction;
L : length of storage bassin;
P : tidal parameter;
P_t : tidal prism (volume of water, entering at flood tide);
q : discharge per unit width;
q_f : fresh water discharge per unit width;
q_L : value of q at $x = L$;
Q_f : fresh water discharge;
t : time;
T : tidal period;
u_0 : maximum flood velocity;
u_1 : velocity in upper layer;
u_2 : velocity in lower layer;
u_f : fresh water velocity;
u_t : rms tidal velocity;
x : longitudinal coordinate;
Δt : time-step;
Δx : mesh-width;
ε : relative density difference;
ρ_1 : density in upper layer;
ρ_2 : density in lower layer;
$\bar{\rho}$: mean density in a cross-section;
τ_b : frictional stress at bottom;
τ_i : frictional stress at interface.

VIII. — References

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