THE STUDY OF DENSITY-STRATIFIED FLOWS UP TO 1945

PART 2

Internal waves and interfacial effects

by J. B. HINWOOD

Senior Lecturer, Department of Mechanical Engineering, Monash University (Australia)

Part I of this history* dealt with the study of nearly horizontal flows with stable interfaces. As in Part I, most of the flows considered in this second part are nearly parallel but here the departures from parallel motion, such as waves and turbulence, are of interest. The study of internal waves is traced from the hydrodynamicists of the nineteenth century, to the oceanographers of the early twentieth century and ultimately to the work of the meteorologists which has continued to the present time. The study of turbulence in stratified fluids in this period centred around the development of a criterion for the growth or decay of turbulence, based on the Richardson number. Other aspects of turbulence considered were the interaction of turbulence and the density stratification both in modifying the fluid's resistance to shear and in producing mixing.

CHAPTER 1. INTRODUCTION

This is the second of three parts surveying the study of density-stratified flows, and treats wave motion and interfacial effects such as mixing, internal shear stress and some aspects of the dynamic instability of statically stable fluids. Part 3 will deal with studies of essentially vertical motions such as convection in cells and plumes and with closely related topics in interfacial stability. As in Part 1 I must stress that this is the work of a civil engineer and the treatment of the meteorological studies in particular is by no means exhaustive.

In Part 1 (Hinwood 1970) the early references to stratified flows were described, showing the origin of the basic concept that variations in density of a fluid subject to gravity could result in motion. The study of steady, nearly horizontal flows was traced through to 1945, at which stage there were available the basic equations of motion in a number of forms, a few exact and some approximate solutions to these equations and a considerable body of data collected in the oceans, in lakes and reservoirs and in the atmosphere, most of which could be explained qualitatively. Quantitative predictions were few since they involved solving intractible equations or utilising inadequate empirical data for resistance coefficients and other quantities.

The study of wave motion did not rely on steady-flow solutions of the equations of motion and hence developed largely independently of the work discussed in Part 1, although the same concepts and fundamental equations were required. In contrast to the work of Part 1 there was quite good communication between workers with different fields of application and so it is possible to trace the study of wave motion chronologically, as will be done in the next section.

The investigations of interfacial stability and breaking internal waves are relevant to the study of internal shear and mixing, but the exchanges of information between these two fields up to 1945 were so few that they will be treated here as separate topics. Chapter 3 deals with studies of several aspects of turbulence in stratified fluids and reveals the limited state of knowledge at the end of 1945.

Since 1945 both wave motion and turbulence studies have been very numerous and significant advances have been achieved in theoretical and applied work, most of the applications being in meteorology until very recently. As in Part 1, only a few references to recent work will be given in a list separate from the main bibliography and no attempt will be made to outline the present state of knowledge.

CHAPTER 2. INTERNAL WAVES

2.1 The first observations

The earliest observations of waves in stratified fluids date back at least to the Romans, as Pliny the Elder (*c* 77) noted that it was well known to divers that a layer of oil on water calms the waves. Franklin made the same observation in 1773 and conducted experiments in a large pond to confirm his impressions. He offered the explanation, "I imagine that the wind, blowing over water thus covered with a film of oil, cannot easily *catch* upon it, so as to raise the first wrinkles...". This explanation leaves unanswered the equivalent question of why the wind can't easily catch upon the surface, and the credit must go to Aitken (1883) for providing the first correct explanation in terms of the resistance to compression of a surface film, which is one of the mechanisms acting. Miles (1960) discussed compression and surface tension effects, both of which are important, and gave a good review of the early investigations. These effects occur at a surface rather than within a fluid which posesses an interface and will not be considered here.

* Cf. *La Houille Blanche*, nº 4-1970, p. 347-359.

Franklin (1762) made the first observation of true internal waves while watching a thick layer of oil floating on a layer of water in the base of a lamp. By swinging the lamp he found interfacial waves were much more easily generated than surface waves, a fact he was unable to explain. Franklin's unexplained observations were of interest to many later workers including Lamb (1879).

The next reported observation of internal waves was that of William Hall of the Aberdeen shipbuilding form of Hall Brothers. In the 1830's the performed experiments related to the design of clipper ships, in which a thin layer of dyed turpentine was added to a tank of water, principally in order to reveal the downward motion of the water near the bows. Hall wrote a private report on this work in about 1840, but it is not available and the best description of this work is in a paper by Cable (1943). Although the report contained sketches of pronounced internal waves, Hall had no idea that such a situation might occur in practice, or that it might increase the drag of the vessel, hence the proposed name "Hall Effect" does not seem as appropriate as "dead water", the older name for this enhanced resistance to motion. Actual observations of dead water may be much older: Goupill (1909) says that dead water had been observed at the mouths of some French rivers for many years although the cause was not understood.

2.2 The nineteenth century hydrodynamicists

Starting in 1847 with Stokes, the study of internal waves was dominated in the nineteenth century by the classical hydrodynamicists. Stokes derived expressions for the velocity potentials of waves at the interface between two finite superposed layers of fluid and obtained the expression for the celerity of infinitessimal progressive waves:

$$c^2 = \frac{g}{m} \, \frac{\rho_2 - \rho_1}{\rho_1 \coth m h_1 - \rho_2 \coth m h_2}$$

where c is the wave celerity, m is the wave number, ρ_1 and ρ_2 are the densities of the upper and lower layers, h_1 and h_2 are their thickness and g is the acceleration of gravity. Kelvin (1871) independently derived a particular form of this equation for wind waves on water. Taking the depths of wind and water as infinite, Stokes's equation may be reduced to:

$$c^2_0 = \frac{g}{m} \, \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}$$

which was obtained directly by Kelvin for infinitessimal waves propagating freely on the air-water interface. Kelvin then derived an expression for waves being driven by a wind of speed v on the surface of still water:

$$c^2 = (\rho_1 + \rho_2) \, c^2_0 - \rho_1 (v - c)^2$$

which is a particular case of a general formula derived sixteen years later by Greehill.

In the same paper Kelvin introduced the technique of imposing infinitessimal sinusoidal disturbances to study the stability of a liquid surface. Rayleigh's (1883) application of Kelvin's procedure to the interface between two statically unstable strata will be described in Part 3, but he also applied the procedure to moving two and three layer statically stable inviscid fluids and to one with an initially exponential density distribution given by $(1/\rho)\,(\partial\rho/\partial z) =$ constant. For such fluids he defined the stability limits under the action of sinusoidal disturbances. He showed that for his case a three dimensional infinitessimal disturbance could be replaced by a two dimensional one, a result which was not recognised to be generally true until the proof of Squire in 1933. These papers of Kelvin and Rayleigh and those of Helmholtz described below founded the subject of hydrodynamic stability and established the principal techniques used for the next seventy years.

Also in 1883, O. Reynolds in his classic paper on the role of viscosity in instability gave a brief comment on the stability of the counterflow situation and stated that there should be a stability criterion. In an experiment he showed the progression from a plane interface, through periodic waves to breaking waves on the interface.

The next decade saw a very rapid growth in the knowledge of internal waves. Webb (1884) obtained a transcendental equation linking the velocity of propagation and the wave length for an n-layered fluid in a rectangular container, which was quoted in detail by Greenhill (1887), this was more a general result than those of Rayleigh, but of the same type. Greenhill, among many other contributions, gave a clearer derivation of Stokes's expression for the wave celerity in a two-layer fluid. He then extended his analysis to the case of uniform relative motion of the strata and found that for sinusoidal waves the velocity of propagation, u, is given by:

$$g\,(\rho_2 - \rho_1) = \rho_2 m\,(v_2 - u)^2 \coth m h_2 + \rho_1 m\,(v_1 - u)^2 \coth m h_1$$

where v_1 and v_2 are the components of the fluid velocity in the direction of wave propagation. The more general case in which the velocity is a known function of the depth was also discussed by Greenhill, as were translational waves in a horizontal cylindrical conduit of arbitrary cross section containing a two-layer fluid. Greenhill increased the generality of his work still further by considering the effect of another fluid property: interfacial surface tension. Utilising the work of Kelvin and Rayleigh, Greenhill showed that a small progressive interfacial wave in an inviscid fluid will be stable if:

$$v < 2\sqrt{gT}\,\frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}\,\sqrt{\rho_2 - \rho_1}$$

where v is the relative velocity of the strata and T the coefficient of surface tension.

Basset's Text "A Treatise on Hydrodynamics" (1888) contained a more direct derivation of Greenhill's expression for the propagation velocity (Art 389), and contained some work on interfacial instability (Art 416). Following this, Burnside (1889) considered more general density distributions and showed that only the free surface wave is irrotational, and Love (1891), using a streamfunction rather than potentials, confirmed Burnside's results using slightly more rigorous mathematics. Whereas Burnside had arrived at a continuous density variation by taking the limit of a number of discrete layers as their thickness became infinitessimal, Love began with a continuous density variation.

The final contributor to this period of development was Helmholtz, who in 1886 gave a qualitative description of the rolling up of the interfacial boundary layer which he regarded as a method of cloud formation, and in 1888, 1889 and 1890 derived expressions for the energy of internal wave motions, and hence obtained stability criteria and rates of growth of waves at the interface of two strata in relative motion. The last papers provided an alternative technique to that of Kelvin and Rayleigh: the energy method, which has been shown to be a special case of a very general method of stability analysis: the direct method of Liapunov. His method was mistrusted by mathematicians because it was feared that the integration over the volume of fluid could hide a local singularity such as an unstable redistribution or accumulation of energy. However such a singularity cannot occur in a continuous viscous fluid and most later workers have confirmed Helmholtz's results, except where his neglect of viscous action caused serious errors.

One of the earliest practical applications of Helmholtz's work came from Wegener (1906), who showed from atmospheric data for cumulus rolls that Helmholtz's equations gave wave lenghts 27 percent too long. Wegener subsequently (1912) corrected his calculations and showed that Helmholtz's predictions were accurate, however Wegener's work involved some personal judgement and it should not be relied upon except to show that the order of magnitude predicted by Helmholtz is correct.

Closely related to these analyses of stability were two sets of descriptive observations made late in the nineteenth century. The first was that of Blasius (1875) who explained the origin of certain types of cloud as being due to interfacial waves generated by the relative motion of two strata of air. The second was that of Forel who described the violent turbulence generated by relative motion at the point where the river Rhone plunges beneath the surface of Lake Geneva. The link between these descriptions and the rigorous but restricted results of hydrodynamical theory was not made untill well into the twentieth century.

2.3 The start of the twentieth century : dead water

The start of the twentieth century saw pure hydrodynamics and descriptive observations replaced by fluid mechanics, the analytical and experimental study of real fluids. In an engineering context Prandtl was the pioneer and in a geophysical context it was V.W. Bjerknes and his pupil V.W. Ekman. The study of waves in stratified fluids reflected this development and swung away from the increasingly complex hydrodynamic analyses into two fields in which experiment could combine with theory, one being the study of internal seiches to be treated in the next subsection and the other being waves in flow past a body. The latter includes waves due to flow past a stationary body — lee waves, to be treated in a later subsection — and waves due to a body moving through a stationary fluid — dead water, to be treated below.

The Norwegian Polar Expedition of 1893-1896 led by Nansen, sailed in the "Fram", a slow beamy vessel easily retarded by the internal waves she produced in the shallow stratification of the Arctic Sea. The term dead water was probably coined on that expedition and was first used in a publication in 1904 by a member of that expedition V.W. Ekman. Meyer (1904) also described the occurrence of dead water in polar seas and at the mouth of the River Congo, but did not propose a mechanism, although in one part of his paper he did suggest that the lower layer might be moving in the opposite direction to the ship. V.W. Ekman (1904) on the other hand performed analyses and experiments which correctly predicted and described the effect.

Ekman's experiments showed that dead water occured when the interfacial waves were out of phase with those on the free surface. These are internal waves of the second mode and have a lower celerity than those of the first mode, which are almost identical with free surface waves in a homogeneous fluid. At ship speeds below this wave celerity, particularly just below it, the internal waves generated rapidly grew to a considerable height, and increased the adverse surface slope near the ship and also increased the velocity of flow past the ship, both effects leading to increased drag. From the point of view of conservation of energy, the energy of the waves had to be supplied by the ship, and near the critical speed a resonant condition existed leading to a maximum rate of extraction of energy. Ekman's theoretical analyses based on inviscid flow agreed well with his experiments in predicting resistance as a function of ship speed. In scaling his results he used various proportionalities, which when combined gave a densimetric Froude number (*) as the constant parameter which is an additional useful result.

Two more hydrodynamic studies which filled in minor details were those of Harrison (1908) and Lamb (1916). Harrisson included the effects of viscosity and was thus able to calculate the rate of energy dissipation which is determined by the energy input and not the other way round. Lamb derived an expression for the velocity potential for the waves in a two layer inviscid fluid subject to a travelling pressure disturbance and obtained good qualitative agreement with Ekman's simpler analysis.

With the end of the sailing ships the subject became of academic interest only, until recently when the practice arose of making extended submarine voyages, often in sharply-stratified rapidly moving waters. However there have been few new developments in the study of dead water since Ekman's work.

2.4 The early twentieth century : seiches

Soon after Ekman's paper on dead water, the study of internal seiches was commenced. A couple of earlier observations had been made of internal seiches in lakes (Thoulet 1894, Watson 1903) but scientific study really began with Watson's (1904) study of Loch Ness.

By observing temperatures on a vertical over a period of time, Watson found that the period of oscillation of the temperature contours, T, was approximately given by:

$$T = 2L \Big/ \left[g \left(\rho_2 - \rho_1 \right) \Big/ \left(\frac{\rho_2}{h_2} + \frac{\rho_1}{h_1} \right) \right]^{1/2}$$

where L is the length of the lake and the other symbols have their usual significance. Watson gave no explanation of the source of this equation, however it may be obtained by assuming the fundamental oscillation which has a wave length of 2 L and using Stoke's equation for wave celerity. Exner (1908 *a, b*) made detailed observations in Lake Wolfgang and from Watson's equation predicted a fundamental period of 24.5 hrs whereas he observed 24 hrs. W. Schmidt (1908) independently derived the same expression but found it predicted periods 5% less than those given by his experiments which he attributed quite reasonably to viscous action.

A more accurate expression for standing internal waves was developed by H. J. Priestley (1909), who retained terms of order (amplitude/wave length)^2. He found that the predicted period of standing waves was greater than with first order theory, but few later workers have made use of his results despite the fact that most observations show slightly longer periods than simple theory predicts, as did those of Schmidt. En 1911, Wedderburn extended Priestley's work by developing a procedure for computing the periods of seiches in basins of irregular cross section, and obtained excellent agreement with his field observations.

The origin of the oscillations was first considered by Watson (1904) who suggested that they were free oscillations following an initial displacement caused by the wind. Wedderburn (1907) tested Watson's idea by blowing air over a small two-layer model and then cutting off the air supply. A solitary wave then formed and travelled back and forth along his tank, but standing waves were apparently not excited. It seems probable that the solitary wave was created beneath a local region of fast air movement, as in Sandström's (1908) experiment with an air jet. A slight tilt of the interface to balance the wind shear on the free surface would have given rise to the fundamental oscillation but must have been too small to see.

A later study of the role of the wind was made by Wedderburn and Williams (1911) who carried out model experiments in tanks of various shapes using layers of paraffin and water and also olive oil and water. Unfortunately the behaviour of the first pair of liquids was affected by interfacial surface tension, and with the second pair a mixed interlayer was formed. The experiments should not be regarded as quantitative but are still of interest, and the periods did agree with Wedderburn's theory. In 1912 Wedderburn found forced oscillations in Loch Earn which correlated with variations in the wind and he also found free oscillations at the fundamental frequency predicted by his theory. His report gives full details of his measuring equipment, data and analyses. In these early studies the movements were inferred from temperature observations at a few sections, and the possibility that the temperature fluctuations were not internal seiches but due to wind pile up could not be ruled out. Birge (1910 *b*) criticised Wedderburn's interpretation of his Loch Ness data and Exner (1910) objected to Halbfass (1909) on this point. Wedderburn's (1908) Loch Garry data is also wide open to this criticism.

H. Pettersson (1920) also found a correlation between the wind component along a fjord and fluctuations in the level of the salinity contours, and found no correlation between the latter and atmospheric pressure fluctuations.

Wedderburn made other studies of particular lakes and Halbfass (1923), Mortimer (1953) and Hutchinson (1957) reported on these and many other less important investigations, and gave excellent reviews of the state of knowledge of internal seiches and waves in lakes. In each of the latter publications the authors explain (following Mortimer 1941,2) that below the thermocline, currents and turbulence in lakes are almost entirely due to internal seiches, hence their importance in lacustrine studies. Hutchinson also refers to an analysis of seiches in a three layer system by Makkaweev (1936) but no other author refers to it and it could not be obtained.

Seiches in the sea were probably first observed by Helland-Hansen and Nansen (1909) and O. Pettersson (1909), who each considered that the oscillations were much less likely to be seiches than tidal motions. Wedderburn (1909) however showed that the period of the oscillations observed by Pettersson corresponded to the fundamental internal seiche. This result and Defant's later calculations of tidal forces (see Defant 1961) cast serious doubts on Pettersson's explanation that the vertical component of the tide-producing forces

(*) The densimetric Froude number, F_D, is defined by :

$$F_D^2 = \frac{v^2}{g\, y\, \Delta\rho/\rho}$$

where v is a typical velocity e.g. the velocity difference between two strata, y a typical depth and Δ_ρ the density difference between strata. In a continuously stratified fluid the gradient Richardson number, R_i, is usually used in place of F_D where :

$$R_i = \frac{g}{\rho}\frac{\partial\rho}{\partial z} \Big/ \left(\frac{\partial u}{\partial z}\right)^2$$

where u and ρ are the velocity and density at any depth z.

caused the internal oscillations. This is a disappointment to me as O. Pettersson's (1913) fascinating paper on the influence of internal tides on the climate near the Baltic depends on this mechanism.
In this paper Pettersson correlated many recorded natural fluctuations and changes, including the Scandinavian population migrations of the last three millennia, with the vertical tide-producing forces. The causal link in this example was the depletion of the northern fishing grounds caused by temperature reductions in the upper water strata, which followed the mixing with the lower, cooler strata caused by high internal waves. The internal waves he supposed were generated in periods of maximum vertical tide producing force. Pettersson's 1935 paper extended the range of phenomena to be explained by this mechanism, but was not convincing.
Zeilon (1913) observed and identified seiches in Gullmar Fiord, and Revelle (1939), Sverdrup (1940) and Munk (1941) found an internal seiche in the Gulf of California. They found that the dominant mode of oscillation had three nodes in the length of the Gulf giving a period close to that of a tidal component. Munk employed W. Schmidt's equations as given by Fjeldstad (1933) but extended them to considering a variable bed slope. The agreement between theory and observation was very good.
An account of recent work off the coast of California was given by La Fond and Cox (1962) in the guise of a review paper, and Defant (1961) gave a good account of observations in the sea up to 1945 with an occasional reference since that time. Recent investigations have applied the existing theories to explain more complicated observations or have made very minor extensions to the theory. Studies of atmospheric seiches or tides are treated in the next section as they differ fundamentally by their essential inclusion of the effects of the earth's rotation and factors peculiar to the atmosphere as a whole.

2.5 Oscillations of planetary scale

Oscillations of the whole atmosphere or ocean are only distantly related to the above seiches, since rotation is of at least equal importance to stratification. Most studies have dealt with the atmosphere but many reduce to oscillations of an equivalent ocean with a free surface, but an ocean which is uniform over the whole of the earth: a spherical annulus.
The semidiurnal oscillation of the atmosphere was first analysed by Laplace (c. 1807) who recognised that it could not be a purely tidal effect as it has a much greater amplitude than the diurnal or lunar components the latter of which must be tidal. Although this led him to suppose that the oscillation is of thermal origin his simplified analysis reduced to the same problem as the one he had solved for tides in the ocean.
Kelvin (1882) pointed out that the above facts showed that the fundamental period of oscillation of the atmosphere must be very close to — in fact within 4 minutes of — 12 hours to have led to the amplification of the effect of the semidiurnal temperature wave. Most later workers started from this fact and sought an appropriate period of free oscillation.
Lamb (1910) improved upon Laplace's analysis by considering an atmosphere in which pressure changes were adiabatic instead of isothermal. He treated in detail the case of a linearly stratified compressible atmosphere above a rotating plane. Chapman (1924) improved upon this model of Lamb's and reconsidered tidal as well as thermal forcing, showing both to be equally important. Lamb re-examined the problem and his work, summarised in Lamb (1932), showed that the infinite number of possible waves is reduced to one mode if the atmosphere is in convective equilibrium. He further extended the analysis by permitting the fluid to move. Solberg (1928) treated oscillations in an incompressible stratified fluid in a rotating reference frame for quite general cases of two and three moving layers.
A totally different mathematical formulation is due to Bjerknes and others (1933) and Solberg (1936). They utilised the linearised equations of motion in the Lagrangian form, and obtained relationships between wave celerity and wave number for various stratifications and other parameters, and set up equations for the velocity distribution. This approach has not been used in the study of planetary oscillations but has been followed up in the study of cyclogenesis.
The tidal origin of the wave was postulated again by O. Pettersson (1934) who criticised earlier analyses which neglected the vertical force, but he merely showed that two events with 12 hours period will give good correlation.
Seeking a period of atmospheric oscillation of 12 hours, Taylor (1929) found one of 10 $\frac{1}{2}$ hours and obtained detailed agreement with observations of the atmospheric wave caused by the explosion of Krakatau. In his 1932 paper he showed that a somewhat more realistic model of the atmosphere failed to give a period of 12 hours, but in 1936 he found that an improved model gave an infinite number of periods although only one for convective equilibrium.
Starting with Taylor's 1936 results, Pekeris (1937) in a definitive paper predicted both the 10 $\frac{1}{2}$ hours and 12 hours free periods and obtained correct phase relationships for a thermally induced semidiurnal oscillation. The semidiurnal wave is a true internal wave in which the outer layers of the ionosphère oscillate with opposite phase to the lower atmosphere. The exhaustive review by Chapman (1951) mentioned other studies and showed that Pekeris had had the last word at that time.

2.6 Remaining theory and observations to 1945

The studies of the previous section were motivated by the need to explain the semidiurnal atmospheric oscillation. This work provided the stimulus for some of the studies in this section and used the results of others. Thus these sections report contemporaneous studies and are only in the broadest sense in chronological order. This section treats the nucleus of the topic of internal waves, starting with the work of the Scandinavian oceanographers who observed internal waves, gave qualitative explanations of their origin and then developed their theory in conjunction with the English applied mathematicians and the German meteorologists. These hydrodynamic studies led to further stability analyses. The study of lee waves is left to the next section as it forms a fairly self contained topic following on from the basic theory.
O. Pettersson's (1909) observations of long period waves have been mentioned in the previous section where it was noted that his explanation in terms of the vertical tidal force is incorrect. Knudsen (1910) also observed long period internal waves, as did Jacobsen (1913) who showed that they correlated with the phases of the moon, but he went no further in explaining their origin. Nansen (1900) and Helland-Hansen and Nansen (1909) observed long period waves and made the tentative suggestion that they were caused by the normal tide of the Atlantic breaking over the submarine ridge which runs from Scotland to Greenland.
Zeilon (1917, 1934) took up this idea and carried out experiments which showed that the normal tidal oscillations could pump an underlying stratum over a submerged barrier from the side on which the interface was higher. Such a process could give rise to an oscillation of period 14 days, which was one of the periods detected by O. Pettersson. Zeilon also noted that short period waves were formed in his experiments. He obtained potential flow solutions for such waves generated in a uniformly oscillating two-layer fluid moving over a bed with either a sinusoidal profile or a low transverse barrier or a section undergoing vertical oscillations.
Helland-Hansen and Nansen noted that years of strong internal wave motion were also stormy years but although the workers on internal seiches have proven wind stress to be the main initiator of motion in lakes, the picture is not as simple in the sea. As noted in an earlier section, H. Pettersson (1920) found a strong correlation between fluctuations of interface depth and wind data but not with atmospheric pressure changes. In this case however the interface fluctuations appear to have been caused by the surface layer being driven first to one end of the basin and then to the other by the wind shear, and were more nearly a succession of steady states than an oscillation. Sandström's (1908) demonstration in which an internal wave was produced by a jet of air was interpreted by Johnson (1919) and others not in terms of wind shear but as the effect of a local reduction in atmospheric pressure. That both mechanisms are capable of producing internal wave trains has been shown by the theoretical studies of Lamb (1916) referred to in Section 2.3 and Takegami (1936) who considered an inviscid two layer fluid with a moving localised shear stress at the surface.
Direct lunar tidal oscillations, although suspected from the outset, were not convincingly identified until the measurements of H. Pet-

tersson and Kullenberg (1933) in the Kattegat although Defant (1927) and others had sometimes found periods of about 12 or 24 hours at a few of their stations for durations of a few days. Pettersson and Kullenberg also found planetary wave periods represented. Seiwell (1937, 1938, 1939, 1942) conclusively proved the existence of lunar tidal periods and also the shorter periods (1941) unreliably indicated by earlier workers. His 1942 paper contains elaborate statistical analyses which show that the lunar and not solar tidal periods are present.

The hydrodynamic studies were continued, primarily by the English applied mathematicians but with important contributions from the Scandinavian oceanographers and the German meteorologists. Harrisson (1908) was the first to include the effects of viscosity in analyses of wave motion. His principal conclusions were that the rate of energy dissipation is proportional to the square of the amplitude and that it is much greater for internal waves out of phase with free surface waves than for those in phase, as is to be expected. He also computed the slightly increased periods of internal waves which result when viscous effects are considered. Jacobsen (1913) computed internal shear stresses from his observations but neglected them in his energy analysis of wave motion in which he independently derived an equation for wave celerity equivalent to Greenhill's.

Lamb (1910) extented the work of Burnside on waves in a continuously stratified inviscid fluid and as has already been mentioned he studied the case of a compressible fluid on a rotating earth. Although this appeared in the fourth edition of his Hydrodynamics in 1916, it was overlooked by other workers on atmospheric tides until the mid 1930's. In 1910 he also briefly looked at waves on a sharp interface. Zeilon (1915) used Burnside's method of dividing a continuously stratified fluid into many thin homogeneous layers which he permitted to have different velocities. He formulated general equations and solved simple examples for waves in such a continuously stratified interlayer between two homogeneous layers. He considered only waves which were long compared to the interlayer thickness, a case relevant to long waves on the ocean thermocline, and found the small corrections from Stocke's formulae for the sharply stratified case.

The first general treatment of internal waves in a compressible fluid was that of V. W. Bjerknes (1916) who considered an inviscid continuously stratified fluid. Using the Lagrangian equations of motion, he determined the density and pressure of a particle as a function of time, which he then inverted to find conditions at any fixed point. He also solved the equivalent equations for an n-layer compressible inviscid fluid.

This work of Bjerknes and a couple of studies in the 1920's do not lie in the main stream of development of the subject and have not been followed up. The later studies referred to are Solberg's (1928) analysis of waves on a sloping interface and Kochin's (1928) highly mathematical derivation of expressions for permanent waves of finite amplitude in a two layer fluid.

On the other hand the very simple derivation by Väisälä (1925) of the period of natural oscillation of a parcel of air in a stratified atmosphere has been very widely used in subsequent analyses and interpretations of data. Väisälä used the frequency to explain fluctuations in the velocity of a series of pilot balloons released from a point into a steady wind. The same frequency was given independently by Brunt in 1927. It is known as the Väisälä frequency or the Brunt-Väisälä frequency.

At this time several reviews appeared; of these Lamb (1932) summarised most of the theory in his usual disjointed style, but the best review was that of V. W. Bjerknes and others (1933) who not only summarised the theory in their excellent fluid mechanics text, but also presented many of the equations in Lagrangian form for the first time.

The Eulerian analysis of internal waves in an incompressible fluid was reviewed in 1933 by Fjeldstad who also presented many new solutions. Fjeldstad examined the stability and wave motions possible in arbitrarily stratified (statically stable), incompressible fluids and obtained solutions for a number of cases including a three-layer system with the density given as the following function of the depth:

$$\rho = \rho_0 \qquad 0 < z < h_1$$

$$\rho = \rho_1 \quad e^{-2\beta(z-h_1)} \qquad h_1 < z < h_1 + h_2$$

$$\rho = \rho_1 \quad e^{-2\beta h_2} \qquad h_1 + h_2 < z < h$$

which may be made to correspond very closely to the stratification in the ocean or in a lake. Fjeldstad also derived formulae for internal waves in the presence of the coriolis force, and in addition explained a procedure for analysing internal wave records previously used by Defant (1932 *a*). In his 1938 paper Fjeldstad gave additional examples showing excellent agreement with observations made in the Herdelfjord near Bergen.

In another quite general analysis of internal waves, Godske, in the first part of his 1935 paper, derived various energy integrals. In contrast to the simplicity of Godske's results, Hylleras (1939), by a variational method, reduced the eigenvalue problem for two and three dimensional oscillations in a stratified sea to the evaluation of a single terrible integral in terms of the boundary conditions; modern computers would enable solutions to be obtained with ease.

While Fjeldstad, Godske and Hylleras were concerned with the applications of internal wave theory, Dubreil-Jacotin was concerned with mathematical rigour. In 1932 he repeated, for a more general stratification, Burnside's proof that internal waves of unchanging form are rotational and showed that in an infinitely deep continuously stratified fluid the waves are Gerstner's rotational waves. In 1934 he obtained an equation for the stream function in a two-dimensional fluid which he solved by successive approximation. A similar equation was later obtained independently by Long (1953). Old observations and stability analyses already mentioned had indicated that internal waves could be produced by the relative motion of two strata, and this was strongly supported by Krümmel (1907), but a quantitative theory was lacking until Taylor's paper of 1914. Taylor used the method of small sinusoidal disturbances, developed by Kelvin and Rayleigh, to obtain stability criteria for small internal waves in fluids with several different density distributions. He did not publish this paper in the open literature until 1931 and this work had little influence on research until that time since when it has had a major effect. In the same year he also published an extension of this theoretical investigation (Taylor 1931 *b*). Prior to 1931 he had already published two papers on interfacial stability. One of these, Taylor (1927 *a*), was a brief description of a qualitative experiment which demonstrated the occurrence of interfacial disturbances and a stability limit, and which for some reason caught people's attention and led to increase interest in this topic. The other paper, Taylor (1927 *b*), dealt with similar but more detailed experiments. In his experiments an undular transition stage was not observed, and the flow changed abruptly from laminar to turbulent. An undular transition stage has usually been observed in similar flow situations e.g. Reynolds (1883), Keulegan (1949) and Macagno and Rouse (1961).

Other analyses of the stability of stratified flows were those of V.W. Bjerknes (1925) for a multilayer compressible fluid, Goldstein (1931) who considered a two layer flow with one or more interlayers and also considered large density differences, and Synge (1933) who examined the stability of a continuously stratified fluid. Although Synge gave an exceptionally clear and precise exposition of small perturbation analysis of hydrodynamic stability with a number of original solutions as worked examples, his paper was published in a lesser known journal and was overlooked for many years.

Hoiland (1944) considered the flow of two linearly stratified layers in a two dimensional duct. He so restricted the problem that it applied only to infinitessimal waves in liquids, although he expressed it in terms of a gas. He solved the eigenvalue problem for the case of two homogeneous layers, obtaining a result in accord with previous work of which he was apparently unaware. Some of his other results are questionable — for example, like Goldstein, he found that for a two-layer inviscid flow there are always unstable waves, but he obtained this result by letting the wave length become arbitrarily small while holding the amplitude constant, thus producing an arbitrarily steep wave. Taylor (1931 *b*) had already shown that for a small enough Richardson number no infinitessimal waves are unstable and this result was later extended by Miles (1967) developing the work of Eliassen, Holland and Rees.

The existence of short period internal waves in the atmosphere due to Kelvin-Helmholtz instability was demonstrated from ground level pressure fluctuations by Goldie (1925), and by Haurwitz (1931 *c*), Jacobs (1937) and earlier workers already listed from observations of cumulus clouds. In his analyses of the stability of internal waves, Haurwitz (1931 *a, b,* 1932) corrected Wegener's calculations of wave length by allowing for the effects of compressibility and condensation of water vapour, and in several examples included coriolis terms.

Linearised stability analyses such as the above indicate only the rate of growth of infinitessimal waves, whereas the rate of growth may decrease once the waves become of finite size (Stuart 1960). The experiments of Defant (1929) on breaking internal waves and the analysis of Rosenhead (1932) on the instability of interfacial waves and their roll up into vortices took the study of interfacial waves beyond the infinitessimal limitation. Another form of instability of a large amplitude internal wave was reported by Jacobsen and Thomsen (1933) who observed a wave in the Straits of Gibraltar which steepened to an internal bore during spring tides.

Quite a different instability to those considered above is the rotational inertial instability of a stratified fluid, such as occurs in frontogenisis. Interesting descriptions and measurements of unstable atmospheric fronts were given by Exner (1923) who also demonstrated similar phenomena qualitatively in a model. The model experiments of Harwood (1945) on the other hand have nothing to do with this problem as they are completely unscaled and almost certainly dominated by poor inlet and outlet conditions. Analytical treatments of this type of instability or relevant to it were given, among others by V.W Bjerknes (1925, 1929), Haurwitz (1931 *b*, 1932) and J. Bjerknes and Godske (1936). The last authors applied the analysis of Godske (1935) who considered internal wave motions in a rotating cylindrical annulus. The studies reported in the previous section, in particular those of Solberg are relevant to this problem too. The use of the pressure tendancy equation for the analysis of frontal motions was proposed by J. Bjerknes (1937), but since it is purely kinematic it is of limited value (it is obtained by forming the time derivative of the hydrostatic equation for pressure). The whole topic of frontogenis was well reviewed in Haurwitz's text of 1941.

From this work outlined in this section it is apparent that meteorologists and oceanographers of the 1930's had quite sophisticated theories available to relate wave periods and celerities and to predict velocity distributions, but because of instrumental limitations and variability of natural systems, data collection and analysis lagged behind. Investigators often failed to do more than identify the dominant frequency present in a data sample, for example Schott (1928) and J. Schmidt (1922, 1929) performed minimal analyses and even Jacobsen and Thomson (1933) using their own data as well as that of Schott and Schmidt performed a rather inconclusive analysis of the periods present in the wave motion of the Straits of Gibraltar. On the other hand Lek and Fjeldstad (1938) using Lek's (1938 *a*) data provided a very convincing confirmation of the predictions of Fjeldstad's (1933) theory, as did Fjeldstad's (1938) observations.

While the oceanographic observations at this time were more numerous and generally preceded comparable atmospheric investigations, the problems raised by both were similar and the hydrodynamic studies described above were as frequently concerned with applications in one field as in the other. The oceanographic information suffered from the limitation of having to make observations by repeatedly lowering a wire to which sampling bottles were attached. This prevented detection of waves of periods less than about an hour although many observers suspected their existence. The neutrally buoyant float unsuccessfully used by O. Pettersson overcame this problem when developed by his son H. Pettersson and Kullenberg (H. Pettersson and Kullenberg 1933, Kullenberg 1932, 1935, Pettersson 1938). The float was first used in the Kattegat and revealed only long period oscillations, but in essentially the same form it is now widely used to detect a broad spectrum of internal waves (La Fond and Cox 1962). Seiwell (1941) was probably the first to obtain useful records of short period internal waves in the sea. The meteorologist on the other hand could make use of ground based observations such as the ground level pressure fluctuations and cloud observations already cited.

Reviews of the knowledge on internal waves more recent than those of the early 1930's have been given by several authors. Prandtl (1949) covered most aspects of the theory of internal waves of all types and gave some data on atmospheric waves. Malone (1951) edited a rather uneven volume treating all aspects of atmospheric motion but stressing observation at the expense of a coherent treatment of the theory. Hutchinson (1957) presented the theory and data on all aspects of internal waves in lakes and Neuman and Pierson (1966) did the same for internal waves in the sea. Yih (1965) treated only the theory, but much more fully than any other recent author.

2.7 Lee waves

The waves generated by an object moving through a stratified fluid were studied by Ekman in 1904, and work on this subject is reviewed in Section 2.3. The corresponding waves generated by a stratified fluid moving past a stationary object were not investigated until some years later. The study of these waves, known as lee waves from their appearance in the lee of mountain barriers, has been left until now to keep this account close to chronological order.

Banded clouds and even temperature fluctuations (e.g. in the Föhn) had been observed in the lee of mountains for many years, but the existence of atmospheric lee waves was not generally recognised until the late 1930's. Since that time they have been studied because of their effect on aircraft. Lee waves in water on the other hand are of much less importance.

The first study of lee waves in the sea was the experimental and analytical investigation of Zeilon (1917, 1934) already mentioned in connection with the origin of internal waves in the sea, in which the flow of an incompressible two layer fluid was studied. Zeilon and later Defant and Helland-Hansen (1938) observed lee waves in the sea, and Hylleras and Romberg (1941) extended the analyses of Hylleras (1939) to cases where the sea bed was no longer plane. Apart from these studies, lee waves have been almost exclusively the preserve of the meteorologist.

The first treatment of the problem was that of Pockels (1913). He assumed that the basic flow was uniform, two-dimensional, inviscid and that the temperature distribution was given by the dry adiabatic lapse rate, and that the small disturbance due to the sinusoidal bed died out exponentially with height. Numerical examples from his solutions agreed well with the incomplete data available to him.

The first analytical treatment to properly consider compressibility of the fluid was that of Raethjen (1929). He first obtained solutions of Bjerknes circulation theorem for flow over a horizontal bed in terms of a series expansion in an imposed perturbation. He then extended this to an analysis of flow over a sinusoidal bed. If he had taken only the first term of his series he could have formed the sum of a number of solutions with different wavelengths which could represent a mountian range. He realised that the upper and upstream boundary conditions are potentially troublesome and imposed an exponential upwards decay on wave amplitudes as Pockels had done.

Queney (1936, 1941) carried out a similar study starting from the linearised equations of motion. He considered coriolis force and the extent of the obstacle and showed for typical atmospheric values that the earth's rotation is unimportant for scales less than 10 km and determines the wave lengths for scales greater than 100 km. His comparisons with large scale atmospheric measurements were interesting but inconclusive for lack of adequate data.

Lyra (1940, 1943) improved the simulation of the atmospheric conditions by employing a polytropic equation of state. For an isothermal atmosphere he solved directly for the waves generated by rectangular steps and ridges instead of using Fourier series methods as the previous workers had done. He computed vertical velocities and Stümke (1940) computed horizontal velocities for his model. Lyra was the first to explain rotors — the eddies with horizontal axes sometimes found below lee waves — by showing that ground level pressure gradients would give rise to flow reversals in the atmospheric boundary layer near the ground.

Prandtl (1949) pointed out that Lyra's model gives amplitudes which increase indefinitely with height, but it is only to be expected that such a crude representation of the atmosphere would be in error to some extent. Prandtl showed that Lyra's theory predicted wavelengths in fair agreement with those observed by Stormer (1939) from sightings over northern Europe of a meteor trail in the stratosphere. Görtler (1941) gave a more convincing confirmation of Lyra's work.

Lyra's work was criticised by many writers in the 1950's because he had applied linearised equations at a step, where slopes were of necessity finite (in fact infinitely large). Subsequent analyses have shown that the error introduced is negligible, and in certain cases is within the normal errors involved in linearisation.

Detailed comparisons of these four analytical models have not been made, and in fact the first three have been ignored by the overwhelming majority of workers. I suspect that this is not because of the merits of Lyra's model, which is in fact not very different to Queney's, but rather because of the dissemination of his findings by Prandtl in his widely read book.

Field observations and wind tunnel studies of the lee waves of

Mt. Fujiyama were carried out by Abe (1932,1941). Although he formed lee waves in his model he had erroneously scaled using Reynolds number and not Froude number so had not reproduced density or velocity variations with height. The first correct model study was performed by Long in 1954 and 1955.

Field data obtained in this period consisted of balloon and glider traverses (e.g. Kempe 1940) and some ground level temperature and cloud observations; the last such study being that of Manley (1945). These observations are described in detail with illustrative photographs in the excellent monograph by Queney and others (1960). Two other studies to lee waves remain to be mentioned: that of Kochin (1938), who obtained potential flow solutions for the interfacial waves generated by a hydrodynamic source on the bed. The second is the waves generated by the oscillations of a flat plate in a stratified fluid studied by Zeilon and by Görtler (1943) which fall in between dead water and lee waves. At one limit of the motion Görtler was able to utilise Lyra's analysis as a check on his experimental results, but he also obtained his own potential flow solutions for the case of very small oscillations.

The knowledge of lee waves at the end of 1945 is contained in the detailed treatment of Queney (1947) and was summarised by Prandtl (1949) and Queney (1948); up to which time there had been little communication between workers in this field. Since then there have been many contributions to the theory which were critically reviewed by Queney and others (1960) and are presented in Yih (1965). Good quantitative date are also available now.

In concluding the topic of internal waves it is useful to realise that while the principal theoretical work on interfacial waves was conducted in the nineteenth century and compressibility was included from about 1910, the study of the dynamic instability of a statically stable fluid was not really started until the 1930's. Furthermore accurate field observations and dynamically scaled model experiments were almost unknown. One consequence of this was that only a weak theoretical basis was available for a deterministic study of mixing processes.

CHAPTER 3. TURBULENCE, MIXING AND SHEAR IN STRATIFIED FLUIDS

3.1 Descriptions of interfacial mixing

Mixing of a stratified fluid caused by internal waves or other disturbances was observed and described years before any quantitative laws were formulated. Even now it is only for simple flows that the onset of mixing can be predicted, and the laws governing the mixing process beyond this initial point are known only for a handful of flow situations.

The early observations, aside from Forel's description of intense turbulence in a lacustrine density underflow, were of mixing in salt wedges. Starting with Franklin in 1761, scientists noticed that the freshwater out-flow entrained the underlying salt water. In 1876, F. L. Ekman described this entrainment and made the important observation that the velocity maximum occured at the same depth as the maximum density gradient, which suggests that the interface is a layer of minimum resistance to shear. This conclusion is in accordance with the statement made by Robert Boyle (1673) that he had often observed strata of water "sliding over one another, in some parts of the sea, especially near the mouths of rivers".

F. L. Ekman and various others in three later papers gave descriptions of entrainment and data for sharply stratified flows. They were O. Pettersson and G. Ekman (1891), F. L. Ekman and O. Pettersson (1892) and O. Pettersson (1894). Less clear descriptions of entrainment in a salt wedge were given by Wheeler (1906), and by Cremers (1908) who gave density profiles for a partially mixed estuary and Rubey (1938) who estimated mixing in a reservoir density current. Other oceanographers and engineers have described the mixing of water strata, but it was not until the work on two-layer flows of Taylor (1927 *b*) and Rosenhead (1932) described in the previous section that the mixing process was studied in detail. While it cannot yet be stated with certainty whether there is always an undular stage prior to mixing or how turbulent mixing first commences, there is a considerable literature on the subject with is discussed in Section 3.3. Following this section studies of the effects of mixing in the interface are reviewed, and in the section below the action of mixing in producing internal shear stresses is considered.

3.2 Shear stress in stratified fluids

In laminar flows the internal shear stress will be unaffected by a density stratification, although the velocity profile will usually change slope if the viscosity varies with the density or if the fluid is accelerated and inertial effects of the density stratification appear. Thus the analysis of Zöppritz (1878) which was based on the use of the molecular viscosity showed no effects arising from the density stratification.

In turbulent flows, shear stresses arise principally through the exchange of momentum between one region of the fluid and another, as a consequence of the turbulence. It is convenient to regard the turbulent shear stresses as the product of the density, the mean velocity gradient and the kinematic eddy viscosity, A*u*, which is also the vertical eddy diffusivity for the longitudinal component of momentum and is a property of the flow and not of the fluid. Some early observations of the effect of a stratification on the resistance to shear have already been described, and others will be discussed below, while in the next section the same processes are discussed with reference to establishing a criterion for the occurence of turbulence.

In his calculation of the spiral velocity trajectories resulting from shear stress and coriolis force, Ekman (1906) used an eddy viscosity and for convenience assumed it to be constant, although he recognised that this was not correct. He observed that the wind stress produced a greater velocity near the water surface when there was a strong density gradient not far beneath the surface and he explained this as a consequence of the reduced eddy viscosity at the interface.

One of the first analyses to consider the effect of atmospheric stability on momentum fluxes was that of J. Bjerknes (1926). In his very simplified treatment he assumed that the eddy viscosity was given by $\Theta/(\gamma_a - \gamma)$ where Θ is the potential temperature, γ_a the adiabatic lapse rate and γ the actual lapse rate at a particular height. This expression at best approximately accounts for the effect of static or convective stability, but obviously makes no allowance for dynamic stability. A similar expression was used by Exner (1927) for the atmosphere, and Fjeldstad (1936) obtained a related expression from an extensive series of measurements in a tidal current:

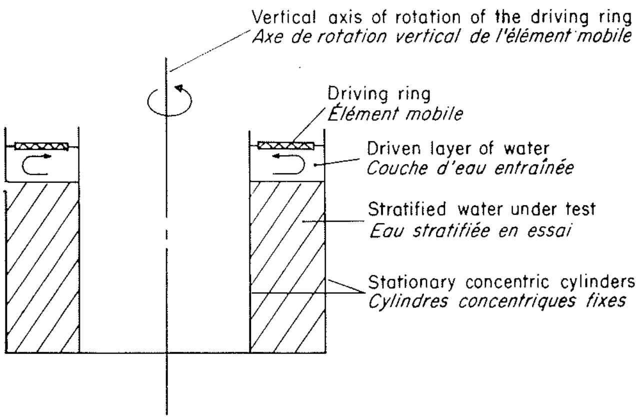
$$Au = f(z) / \left(1 + \frac{a}{\rho} \frac{\partial \rho}{\partial z} \right)$$

where $f(z)$ is a function only of the depth and a is a positive constant.

Jacobsen (1913) made a series of systematic measurements in strongly stratified waters near Denmark and by simplifying the equations of conservation of mass and momentum he was able to evaluate the eddy viscosity, A*u*. He also roughly computed the vertical eddy diffusivity of salt, A*m*, and found it to be about one tenth of A*u*. Jacobsen (1930) obtained similar results from a later series of observations and he explained the difference between the values of A*u* and A*m* by assuming that the turbulent eddies rapidly lost their momentum but were restored to their original position by gravitational forces before they could exchange their other properties by mixing on a molecular level.

Support for Jacobsen's explanation was provided by experimental and field studies of turbulence in stratified flows. Prandtl and Reichardt (1934) determined velocity correlation coefficients using the Göttingen wind tunnel, and showed that turbulence in a stratified flow was anisotropic. The Göttingen wind tunnel could be heated or cooled at the top or bottom and was used for precise experiments on many stratified flow situations, particulary simple parallel flows.

The effect on momentum transfer across an interface of both the velocity and the temperature gradient appears to have been examined first by Hesselberg (1926). From his mixing length analysis he concluded that for cases of relevance to atmospheric motions the additional momentum transfer resulting from the temperature gradient will be two orders of magnitude less than the velocity gradient effect — the assumption made by almost all workers since that time. That this conclusion is not generally correct, even if wall shear is allowed for, is shown by the data of Pritchard (1956) for flow in an estuary where both terms are significant at the interfacial region, and by the analysis of Priestley and Swinbank (1947). Other systematic measurements were those of Durst (1933 *b*) who showed that inversions reduce turbulence, and Suda (1932, 1936) who measured velocity profiles in a rapidly moving sharply-stratified



1/ Apparatus of H. Pettersson and H.W. Malmheden (1935) for measuring eddy viscosity in density-stratified water.

Appareil de H. Pettersson et H.W. Malmheden (1935) pour la détermination de la viscosité tourbillonnaire dans l'eau avec stratification de densité.

ocean current and computed eddy viscosities and diffusivities. He found that although the shear stress did not vary greatly with depth, the energy dissipation attained a maximum at the interface between the strata. Hellström (1940) found that by using a thin interlayer of low viscosity his computed velocity profiles matched those in a wind-induced current in a stratified fluid; such a structure results from suppression of turbulence and hence reduction of the Reynolds stresses in a strong density gradient. Thornthwaite and Kaser (1943) measured numerous wind profiles and showed that stability had a marked effect but as they did not measure the vertical temperature profiles their data are of little value.

In some of the field studies empirical formulae were obtained, and these give a clearer idea of the dependence of the eddy viscosity and related quantities on the stratification. For example in their comprehensive study Rossby and Montgomery (1935) found that the mixing length, l, satisfied the relationship:

$$l = kz/\sqrt{1 + cR_i}$$

where R_i is the Richardson number and k and c are constants. Sverdrup (1936) amended their derivation by assuming a more realistic temperature profile near the ground and showed that it fitted his profiles measured in the atmosphere and the ocean (Sverdrup, 1936, 38, 39), but Holzman (1943) found that the constant c varied and proposed instead:

$$l = kz\sqrt{1 - c^1R_i}$$

where c^1 is a constant. Holzman's formula shows that turbulence disappears at a sufficiently high value of the Richardson number. Deacon (1949) found Holzman's formula fitted data from a number of studies but these formulae have not often been used in recent investigations.

Much earlier than these authors, V. W. Ekman (1906) had devised a simple experiment to determine the effect of the stratification on the eddy viscosity but was unsuccessful because of faults in his apparatus. His apparatus consisted of a rough-bottomed, smooth-walled glass cylinder, filled with a stratified solution and set rotating at a constant speed. The fluid near the bottom was brought into motion by shear, and ultimately this motion was transmitted to a vane at the surface. The eddy viscosity could be computed from the time taken from the start of motion of the cylinder for the vane to start rotating; it was assumed that secondary circulations were unimportant. H. Pettersson (1931) repeated and extended these experiments and showed that the computed eddy viscosity decreased as the density gradient increased. He also described the onset of general instability and overturning which occured at high rates of rotation. That secondary circulations were present was shown by Pettersson, so that the values of eddy viscosity are not applicable to any other situation, but they do give a gross measure of the rate of momentum transfer in the given apparatus.

To improve on this work, H. Pettersson and Malmheden (1935) redesigned the apparatus to the form shown in Figure 1. The steadily rotating ring at the top induced in the upper water layer a rotation about the vertical axis and a secondary circulation. The latter opposed any tendency for circulations to arise in the lower layers, which rotated solely about the vertical axis. They showed the sudden onset of turbulence already shown by Taylor reduced momentum transfer across an interface and the consequent tendency for momentum to be concentrated in the driven layer, particularly while the fluid was being accelerated — a phenomenon later studied analytically by Rossby (1951). Inexplicably these papers have been completely overlooked. An apparatus similar to Pettersson's original one was used by Kato and Phillips (1969) who did not detect secondary flows at the high rates of mixing they used.

3.3 Energy considerations in turbulence and mixing

Neumann (c. 1835) had argued that a heavy layer above a light layer would be unstable but did not give a rigorous proof. Thus the first analyses of turbulence and mixing in which energy was considered were the macroscopic studies of mixing carried out in the first part of the twentieth century although the work of Rayleigh and Helmoltz described in Section 2.2 was in many respects more advanced than this later work. The potential energy change which would result if a stratified body of water were completely mixed was calculated by a number of investigators. In order to calculate the energy changes accurately it is necessary to allow for the effects of compressibility of water, and this was first done for fresh water by Groll (1903, 4). Utilising Groll's data, Birge (1910 *a*) tabulated the work done in mixing a cubic metre of fresh water with a constant temperature gradient of unit magnitude between its upper and lower faces. From this analysis he obtained quantitative estimates of the difficulty of mixing stratified water at various temperatures. Based on their own data for sea water (1915 *a*), Hesselberg and Sverdrup carried out similar analyses (1915 *b*) as did Atkins (1925). The conclusions that may be drawn from these calculations are that one should not confuse temperature gradients with density gradients and that the compressibility of water is often not negligible.

Other macroscopic energy calculations are those cited by Hutchinson (1957) for the mixing of whole strata in lakes. In some of these investigations, in particular Hutchinson (1937), the energy calculations have been used to arrive at estimates of the time required to mix the given body of water. Such a procedure appears to be quite unreliable in view of the low and unknown efficiency of the mixing process which itself depends upon the stratification.

Small scale, or strictly speaking infinitessimal scale studies of the energy involved in mixing a stratified fluid form the most important group of investigations of turbulence and mixing of stratified fluids up to 1945. The first such calculations were made by W. Schmidt (1918), but L. F. Richardson in 1920 pointed out an error in Schmidt's work and presented his own calculations.

To obtain Richardson's criterion for the growth or decay of turbulence, consider a fluid of density ρ and velocity u, in which u and ρ are continuous functions of the height z, and in which there is only one element of turbulence. From considerations of energy conservation and after discarding terms he found to be small, Richardson proposed that the kinetic energy of the turbulent eddy would decrease if the rate at which energy was extracted from the mean motion by the Reynolds stresses was less than the rate at which work was done against the gravitational forces. By using an eddy viscosity and diffusivity, the ratio of these quantities is found to be:

$$\frac{g \dfrac{\partial \bar{\rho}}{\partial z} Am}{\rho \left(\dfrac{\partial \bar{u}}{\partial z}\right)^2 Au}$$

where Am is the vertical diffusivity of mass (or heat in the atmospheric case), Au is the vertical eddy viscosity and the bars denote the appropriate averages at a point. The quantity:

$$\frac{g}{\rho}\frac{\partial\bar{\rho}}{\partial z}\Big/\left(\frac{\partial\bar{u}}{\partial z}\right)^2$$

is the gradient form of the Richardson number and is denoted by the symbole R_i. Richardson's criterion thus becomes:

$R_i > Au/Am$ turbulence is decreased by the stratification and hence turbulent mixing is unimportant.

$R_i < Au/Am$ turbulence is increased by the stratification and hence turbulent mixing will occur.

A test of this criterion was made by Richardson (1925) who obtained data showing that the criterion was in the range of $R_i \lesssim 0.5$ to 1.0. There was a considerable body of data showing that in homogeneous flows Au/Am was usually within the range of 1.0 to 2.0 and although much larger values had been obtained, Richardson's test was regarded as supporting his analysis to at least order of magnitude accuracy.

Prandtl in 1929 derived a criterion which differed only in that it was larger by a factor or two. To test his theory he carried out some laboratory experiments and obtained his predicted criterion $R_i \lesssim 2.0$ with some experimental uncertainty. The error in his analysis was pointed out by Taylor (1931 *b*) and was corrected by Prandtl in 1932 and 1933.

This correction did not settle all controversy over the criterion for the growth or decay of turbulent mixing since the previously mentioned studies of the stability of internal waves of Taylor (1931 *a*) and Goldstein (1931) had indicated that the criterion should be $R_i \lesssim 0.25$. An additional complication was the fact that while Richardson's energy analysis had yielded the criterion $R_i \lesssim Au/Am$ this ratio could not be assumed equal to one as he and others mentioned in the previous section had shown.

Data obtained at this time did little to clarify the choice of the correct value of the critical Richardson number. Taylor obtained partial confirmation of his energy criterion by analysing the data of Jacobsen (1913) for which he showed that the flow had been turbulent at all times and that as he predicted $R_i < Au/Am$ at all times for a range of Au/Am from 5 to 50 in the strongly stratified waters. Prandtl in his 1932 paper stated that available data supported a value between 0.25 and 0.5, but Durst (1933 *a*) found for a natural wind that the critical value was very close to 1.0 although he only checked this criterion as an afterthought and then only on one of his many sets of data. Holzman's (1943) empirical formula for mixing length provided a systematic basis for extrapolating to the condition of no turbulence and, using Deacon's (1949) value of 7 for the constant, it gave a value of 0.14 for the critical Richardson number.

The investigation which came the closest to resolving the controversy was that of Prandtl's student Schlichting in 1935. From measurements in a continuously stratified boundary layer in the Gottingen wind tunnel, Schlichting obtained values of the critical Richardson number lying between 0.029 and 0.041. The actual value depended upon an additional parameter, the Reynolds number R_e. In the same paper he presented the clearest available derivation of the Richardson number as the sole criterion and then showed how the analysis could be improved to show the dependence of the critical Richardson number on the Reynolds number. Later workers have confirmed that the critical Richardson number depends on the nature of the flow, including the Reynolds number and the boundary roughness in many cases.

A fresh analysis was made by Ertel (1929, 1939-1944) who divided the numerator of the Richardson number into an irreversible part and a reversible part which is a purely thermodynamic quantity resulting from the compressibility of the fluid. Prandtl (1944 *a*) published a rebuttal of Ertel's analysis and further criticised its usefulness in another short paper (Prandtl 1944 *b*). Ertel also suggested that not all factors were taken into account in the usual form of the Richardson number but it was not until 1947 that C.H.B. Priestley and Swinbank showed that there was an additional term caused by the correlation of buoyancy fluctuations with velocity fluctuations.

An interesting application of a Richardson type of criterion to a reservoir underflow was made by Monish (1938) who based his work on Prandtl's (1929) analysis. For this two-layer flow he chose a length scale proportional to ν/u, where ν is the kinematic viscosity, which transformed the Richardson number into a densimetric Reynolds number, a parameter utilised later by Keulegan (1949) in his studies of internal stability. For the velocity he substituted in terms of the depth and the discharge, Q, and for the density difference in terms of the silt concentration, n, and obtained the following simple relationship for a given underflow:

$$n/Q = \text{constant}.$$

Despite some outrageous assumptions in his derivation, the formula fitted data from two different reservoirs. A thorough review of the work prior to the publication of the papers of Schlichting and Ertel was given by Brunt (1934), and Schlichting (1935) also reviewed previous investigations and gave a comprehensive list of references. The exact value of the criterion is no longer regarded as such an important point, present emphasis being in the amount of mixing after the criterion (if it exists!) has been passed.

3.4 The role of mixing in modifying a density stratification

So far in this section we have followed the attempts to develop a criterion for turbulence to penetrate a density gradient and hence cause mixing, and have examined the effects of the density stratification on the turbulence as shown by its effects on the shear stress. The study of the action of the turbulence on the density stratification complements the latter studies and extends the work on a criterion for mixing the cases in which significant mixing occurs. The quantitative results obtained by Schlichting (1935) for no mixing were of the form $f(R_e, R_i) = 0$. More recent studies have shown that similar functional relationships hold for a given degree of mixing, among the first being the investigations of Keulegan (1949) and Rouse and Dodu (1955). With the exception of the work of Monish on turbid underflows and Hutchinson (1937) on entrainment of a stagnant lower stratum in a lake, both of which involved many approximations, no such studies were made up to 1945. Instead investigations of mixing were almost entirely qualitative and were concerned with determining the mechanisms of processes which were of importance to the understanding of various mean flow situations.

The principal process studied was the generation of the thermocline in lakes and in the sea. In 1892, Forel showed from his studies of Lake Geneva that surface heating and cooling on a seasonal time scale led to thermal overturn and the formation of a steep density gradient at the bottom of the overturning layer. In the simplest case observed by Forel, the surface water cooled in winter until it became denser than the underlying water, when it subsided to form the dense, cold bottom water of the lake which by the end of winter may occupy the full depth of the lake. Summer heating then affected the surface layers first, and a thermocline was formed which slowly descended. Forel described the less common process which is the mixing of very cold (< 4°) water from shallow areas with warmer water (5° - 6°) from the surface of deep areas of the lake. At the zone of contact, mixing leads to the formation of water of maximum density which then flows down the slope to the bottom of the lake. Diurnal heating, cooling and overturn were not recognised until a few years later. Hutchinson (1957) described other thermal regimes and presented a review of literature dealing with classification of lakes according to their thermal regime but such details are not relevant to the fluid mechanics of density stratified flows.

Forel also recognised that the stirring action of wind waves assisted in mixing the upper layer, and Wedderburn's (1907) experiments on a model lake showed that surface heating with mixing from wind waves would generate a thermocline, even with no thermal overturn. From qualitative model experiments and observations of actual lakes, Kindle (1927) showed that action of wind waves was responsible for the sharp upper limit of the thermocline and Atkins (1924) also noted the effects of waves in mixing the upper layers. The observations of Atkins in the English Channel showed that sufficiently strong and prolonged wind and wave action could eliminate the thermocline by mixing the waters over the whole depth.

One other process leading to a sharp interface, but not related to the thermocline, is the flow of one body of fluid over another. The studies of dynamic instability of such flows and the counteracting action of the stratification in inhibiting vertical turbulence have already been discussed. It is evident that the latter effect will tend to maintain a steep density gradient once one has been formed, as was noticed by Pliny, Boyle, F. L. Ekman and later many others including Kuenan (1937). The enhancement or sharpening of a density gradient by stretching the interfacial layers was not specifically mentioned until after 1945 although it was almost certainly known to all experimenters.

Although vertical turbulence is inhibited, lateral turbulence is not greatly affected by a stratification and hence lateral mixing will persist, even in a steep density gradient, and may tend to form homogeneous strata above and below the steepest density gradient. Some observations of the effect of vertical stratification on horizontal mixing in the sea have been made by Parr (1936) and by Montgomery (1938) who followed Rossby and others in considering isentropic rather than horizontal surfaces.

CHAPTER 4. FINAL COMMENT

In contrast to the studies reported in Part I, the study of internal waves has followed a definite sequence with most authors utilising the work of their predecessors. As a consequence the theory of internal waves was well advanced by the end of 1945. A possible reason for this is the leading role of the English hydrodynamicists in developing the theory, avoiding the fragmenting effect of many languages and many fields of application. Even the fact that important simplifications were usually possible had not restricted the development of the theory, thus by 1910 Harrison had included viscous action, Lamb had considered waves in a compressible stratified fluid and V.W. Ekman had solved the case of waves generated by a moving object. Over the next ten years solutions were obtained for waves with difficult geometric boundary conditions. For example Wedderburn computed seiches in basins of irregular cross section, Lamb treated waves in a rotating atmosphere on a sphere and Zeilon solved the case of an oscillating flow above a sinusoidal bed. The comprehensive and original treatises of Bjerknes and others, Fjeldstad and Lamb in the early 1930's marked the end of the predominance of this type of study.

The studies of stability which had started with Kelvin, Rayleigh, Helmholtz and later L.F. Richardson now proliferated with major contributions from Taylor, Synge and Schlichting. Most of these studies have some relevance to the inception of mixing but no proper analysis beyond the point of inception of mixing was conducted up to 1945. The study of mixing and its effects on the shear stress and density profiles was mainly the work of the Swedish oceanographers, one of the most interesting of these studies being H. Pettersson's experimental investigation of shear stress in a multilayer fluid. (This line was written before the work of Kato and Phillips was published, and it is unfortunate that work as useful as Pettersson's, like the stability analysis of Synge and Taylor's essay for the Adams prize, was not published immediately in a more widely read journal).

ACKNOWLEDGEMENT

This investigation had its origin in an essay commenced in 1963 while the author was studying under Professors Rouse and Macagno at the Institute of Hydraulic Research, Iowa City.

BIBLIOGRAPHY

ABE, M. (1932), The formation of cloud by the obstruction of Mount Fuji, *Geophy. Mag.* (Tokyo), 6, 1-10.

ABE, M. (1941), Mountain clouds, their forms and convected air current, Part II, *Bul. Central Met. Observatory,* Japan, 7 (3), 93-145.

AITKEN, J. (1883). On the effect of oil on a stormy sea, *Proc. Roy. Soc. Edinb.,* 12, 56-75.

ATKINS, W.R.G. (1924), On the vertical mixing of sea water and its importance for the algal plankton, *J. Mar. Biol. Assoc.,* new ser., 13. 319-324.

ATKINS, W.R.G. (1925), On the thermal stratification of sea water and its importance for the algal plankton, *J. Mar. Biol. Assoc.,* new ser., 13, 693-9.

BASSET, A.B. (1888), A treatise on hydrodynamics, vol. 2, Deighton, *Bell & Co.* (1888), Dover, N.Y. (1961).

BIRGE, E.A. (1910 *a*), An unregarded factor in lake temperature, *Trans. Wisconsin Acad. Sci. Arts Letters,* 16, 989-1004.

BIRGE, E.A. (1910 *b*), On the evidence for temperature seiches, *Trans. Wisc. Acad. Sci. Arts Letters,* 16, 1005-16.

BJERKNES, J. (1926), Diagnostic and prognostic application of mountain observations, *Geophys. Publ.* (Oslo), 3 (6), 1-35.

BJERKNES, J. (1937), Theorie der aussertropischen Zyklonenbildung, *Meteorol. Z.,* 54, 462-6.

BJERKNES, J. and C.L. GODSKE (1936), On the theory of cyclone formation at extratropical fronts, *Astrophys. Norveg.,* 1, 199-235.

BJERKNES, V.W. (1916), Über Wellenbewegungen in kompressiblen, schweren Flüssigkeiten, *Abhand. math-phys. Klasse Sächs. Akad. Wiss.,* 35, 37-65.

BJERKNES, V.W. (1921), On the dynamics of the circular vortices with applications to the atmospheric vortex and wave motions, Christiania (Oslo Universitet), *Geophys. Publ.,* 2, (4).

BJERKNES, V.W. (1925). Polar-front meteorology, *Quart. J. Roy. Met. Soc.,* 51, 261-8.

BJERKNES, V.W. (1929), Über die hydrodynamischen Gleichungen in Lagrangescher und Eulerscher Form und ihre Linearisierung für das Studium kleiner Störungen, Christiania (Oslo Universitet), *Geofys. Publ.,* 5 (11).

BJERKNES, V.W., J. BJERKNES, H. SOLBERG and T. BERGERON (1933), Hydrodynamique Physique, (transl. of German edition of 1933), vol. 23, Recueil des conférences - Rapports de documentation sur la physique, Paris (1934), 566 pp.

BLASIUS, W., (1875), Storms : Their Nature, Classification and Laws, *Porter and Coates,* Philadelphia, 342 pp.

BOYLE. R., (1673), Works, *E. Flesher,* Oxford, pp. 764-781, Observations and experiments about the saltness of the sea.

BRUNT, D. (1927), The period of simple vertical oscillations in the atmosphere, *Quart. J. Roy. Met. Soc.,* 53, 30-32.

BRUNT., D., (1934), Physical and Dynamical Meteorology, *Cambridge U. Press,* 1934.

BURNSIDE, W., (1889), On the small wave-motions of a heterogeneous fluid under gravity, *Proc. London Math. Soc.,* ser. 1, 20: 392-7.

CABLE, B., (1943), The world's first clipper, *Mariner's Mirror,* 29 (2), 66-85, in particular pp. 73-4.

CHAPMAN, S., (1924), The semi diurnal oscillation of the atmosphere, *Quart. J. Roy. Met. Soc.,* 50, 166-195.

CREMERS, J.J.C., (1908), Invloed van het verschil in soortelijk gewicht van zout en zoet water op strooming en verplaatsing van vaste stoffen in benedenrivieren, *De Ingenieur,* Jaarg., 23, 946-951. Also published as Einfluss des verschiedenen spezifischen Gewichts von Salz- und Süsswasser auf die Strömungen und die Ablagerung von festen Stoffen in Tidesströmen, *Ann. Hydrog. u. Mar. Meteor.,* (1909), 271-7. Summaries in Österreiche Wochenschrift für den offentlichen Bundesdienst (1909), 235-6, and in *Inst. Civil Engrs. London, Proc.,* 176 (II), 373-4.

DEFANT, A., (1929), Dynamische Ozeanographie, Einführung in die Geophysik, vol. 3, *Springer,* Berlin, 222 pp.

DEFANT, A. (1927), Bericht über kurzperiodische Schwankungen von Temperatur und Salzgehalt in den obersten Wasserschicten des Ozeans, *Z. Gesellsch. Erdkunde Berlin,* (1927), 286-300.

DEFANT, A., (1932 *a*), Über die wissenschaftlichen Aufgaben und Ergebnisse der Expedition, *Berichte der Deutsch., Atlantischen Exped. auf Meteor,* 1925-7, Berlin, 359-369.

DEFANT, A., (1932 *b*), Beiträge zur theoretische Limnologie, *Beitr. Freien Atmos.,* 29, 143-150.

DEFANT, A. (1933), Gedenken über interne Gezeitenwellen, James Johnstone Memorial Volume, ed. *R.J. Daniel,* publ. University of Liverpool, Liverpool (1935), pp. 310-315.

DEFANT, A. and B. HELLAND-HANSEN (1938), Bericht über die ozeanographischen Untersuchungen im zentralen und östlichen Teil des Nordatlantisden Ozeans im Fruhsommer 1938, *Abhand. Preuss. Akad. Wiss.,* Phys. - Math. Kl., n° 5, 1-28.

DUBREIL-JACOTIN, M.L., (1932), Sur les ondes de type permanent dans les liquides hétérogènes, *Atti Acc. Lincei,* Rend. Cl. Sci. Fis. Mat. Nat., ser. 6, 15, 814-9.

DUBREIL-JACOTIN, M.L., (1934), Sur les théorèmes d'existence relatifs aux ondes permanentes périodiques à deux dimensions dans les liquides hétérogènes, *J. de Math.*, 13, 217-291.

DURST, C.S. (1933 *a*), The breakdown of steep wind gradients in inversions, *Quart. J. Roy. Met. Soc.*, London 59, 131-6.

DURST, C.S. (1933 *b*), Notes on the variations in the structure of wind over different surfaces, *Quart. J. Roy. Met. Soc.*, 59, 361-372.

EKMAN, F.L. (1876), On the general causes of the ocean currents, *Nova Acta Reg. Soc. Sci.*, Upsala, ser. 3, 10 (6), 1-52.

EKMAN, F.L. and O. PETTERSSON (1892), Den Svenska hydrografiska expeditionen ar 1877 under ledning af F.L. Ekman, *K. Svenska Vet., - Akad. Handl.* Stockholm, Ny F., 25 (1), 1-63.

EKMAN, V.W. (1904), On dead-water, Sci. Res. Norwegian Polar Exped. 1893-6, *ed. F. Nansen, Longmans, Green & Co.*, London, 5 (sec. 15), 1-152.

EKMAN, V.W. (1906), Beiträge zur Theorie der Meeresströmungen, *Ann. Hydrog. Mar. Meteor.*, 12, 423-430, 472-484, 527-540, 566-583.

ERTEL, H. (1929), Zur Energetik atmosphärischen Luftsäulen, *Meteorol. Z.*, 46, 10-16.

ERTEL, H. (1939), Thermodynamische Begründung des Richardsonchen Turbulenzkriteriums, *Meteorol. Z.*, 56, 109-111.

ERTEL, H. (1943), Die Hydrothermischen Grundgleichungen turbulenter Luftströmungen, *Meteorol. Z.*, 60, 289-295.

ERTEL, H. (1944), Zur statischen Theorie des vertikalen Turbulenz Wärmestroms, *Meteorol. Z.*, 61, 207-9.

EXNER, F.M. (1908 *a*), Über eigentümliche Temperaturschwankungen von eintagiger Periode im Wolfgangsee., Sitzungsber. Akad. Wissen. Wien, *Math. -Naturwiss Kl.*, pt. IIa, 117:9-26.

EXNER, F.M. (1908 *b*), Ergebnisse einiger Temperaturregistrierungen im Wolfgansee, Sitzungsber. Akad. Wissen. Wien, *Math.-Naturwiss. Kl.*, pt. II *a*, 117, 1295-1315.

EXNER, F.M. (1910), Zur Frage der Temperaturseiches, *Petermanns Geogr. Mitt.*, 56, 139.

EXNER, F.M. (1913), Über Luftdruckschwankungen in der Höhe und am Erdboden, *Meteorol. Z.*, 30, 429-436.

EXNER, F.M. (1923), Über die Bildung von Windbosen und Zyklonen. Sitzungsber. Akad. Wiss. Wien, *Math.-Naturwiss. Kl.*, pt. II *a*, 132, 1-16.

EXNER, F.M. (1927), Abhängigkeit des Turbulenzfactors der Winde von der vertikalen Temperaturverteilung, Sitzungsber. Akad. Wiss. Wien, *Math.-Naturwiss. Kl.*, pt. II *a*, 136, 453.

FJELDSTAD, J.E. (1933), Interne Wellen, *Norske Vidensk. Akad. Geofys. Publ.*, (Oslo), 10 (6), 1-53.

FJELDSTAD, J.E. (1935), Internal Waves, I.G.G. Union, *Assoc. Phys. Oceanog.*, Proc.-Verb., n.2, Liverpool, England, English abstract of Fjeldstad (1933).

FJELDSTAD, J.E. (1936), Results of tidal observations, Norwegian North Polar Expedition with the 'Maud' 1918-25, *Sci. Res.*, 4 (4), 1-88.

FJELDSTAD, J.E. (1938), Interne Wellen, 15th Int. Geogr. Cong., Amsterdam, 1938, sec. II b, 12, 49-55.

FICKER, H. von & B. DE RUDDER (1943), Föhn und Föhnwirkungen, *Becker and Erler,* Leipzig, 111 pp.

FOREL, F.A. (1892), Le Léman, Lausanne, vol. I, 540 pp., 1892; vol. 2, 651 pp., 1895; vol. 3, 715 pp., 1904.

FRANKLIN, B. (1761), Complete Works, London, paper on the salt wedge etc., 2, 107.

FRANKLIN, B. (1762), Complete Works, London, paper on observations of internal waves, 2, 142.

FRANKLIN, B. (1773), Complete Works, London, Correspondence on the calming effect of oil on water-waves, 2, 144.

GODSKE, C.L. (1935), A simplified treatment of some fluid oscillations, *Astrophys.*, Norveg., 1, 169-197.

GOLDIE, A.H.R. (1925), Waves at an approximately horizontal surface of discontinuity in the atmosphere, *Quart. J. Roy. Met. Soc.*, 51, 239-246.

GOLDSTEIN, S. (1931), On the stability of superposed streams of fluid of different densities, *Proc. Roy. Soc.*, London, ser. A, 132, 524-548.

GÖRTLER, H von (1941), Neuere Beiträge zur Dynamik atmosphärischer und ozeanischer Stromüngen, *Die Naturwissensch.*, 32, 473-9.

GÖRTLER, H. von (1943), Über eine Schwingungserscheinung in Flüssigkeiten mit stabiler Dichteschichtung, *Z. Ang. Math. Mech.*, 23, 65-71.

GOUPIL, (1909), Sur le phénomène de l'eau morte, *Ann. Ponts Chaussées*, ser. 8, 37, 139-141.

GREENHILL, A.G. (1887), Wave motion in hydrodynamics, *Amer. J. Math.*, 9, 62-112.

GROLL, M. (1903-4), Der Öschinesee, *Jahresber. Geogr. Ges.*, Bern, 19, 1-78.

HALBFASS, W. (1909-10), Zur Frage der Temperaturseiches, *Petermanns Geogr. Mitt.*, 55, 364-5, 56, 139.

HALBFASS, W. (1910), Ergebnisse neuerer simultaner Temperaturmessungen in einigen tiefen Seen Europas, *Petermanns Mitt. Justus Perthes. Geogr. Anst.*, 56, 59-64.

HALBFASS, W. (1923), Grundzüge eine vergleichenden Seenkunde, *Borntraeger,* Berlin, 354 pp.

HARRISON, W.J. (1908), The influence of viscosity on the oscillations of superposed fluids, *Proc. London Math. Soc.*, ser. 2, 6, 396-405.

HAURWITZ, B. (1931 *a*), Zur Theorie der Wellenbewegung in Luft und Wasser, *Veröff. Geophys. Inst. Leipzig Univ.*, 5 (1), 1-106.

HAURWITZ, B. (1931 *b*), Über die Wellenlänge von Luftwogen, part I, *Gerlands Beitr. Geophys.*, 34, 213-232.

HAURWITZ, B. (1931 *c*), *Meteorol. Z.*, 48, 483.

HAURWITZ, B. (1932), Über die Wellenlänge von Luftwogen, part II, *Gerlands Beitr. Geophys.*, 35, 16-24.

HAURWITZ, B. (1941), Dynamic Meteorology, *McGraw-Hill,* New York, 365 pp.

HARWOOD, W.A. (1945), Experiments on frontal waves, *Quart. J. Roy. Met. Soc.*, 71, 377-383.

HELLAND-HANSEN, B. and F. NANSEN (1909), The Norwegian Sea, Report on Norwegian Fishery and Marine Investigations, Christiania, 2 (2), 1-390, maps.

HELLAND-HANSEN, B. (1933), The Sognefjord Section; Oceanographic observations in the northernmost part of the North Sea and the southern part of the Norwegian Sea, James Johnstone Memorial Volume, *ed. R.J. Daniel,* publ. University of Liverpool, Liverpool, 1935, pp. 257-274.

HELLSTROM, B. (1940), Wind effect on lakes and rivers, *K. Tekniska Hogskola Anhandl.*, Stockholm, 26, 1-190 pp. Also in *Ingen. Vitensk. Akad. Handl.*, 1941, 158.

HELMHOLTZ, H. von (1886), Wolken und Gewitterbildung, *Charlottenburg Phys.-Tech. Reichsanstalt*, 5, 96-7. Also in *Wiss. Abhandlungen*, 3, 286-287.

HELMHOLTZ, H. von (1888), Über atmosphärische Bewegungen, Part I, *Sitzungsber. Akad. Wiss.* Berlin. 1888, 647-663; Part II, 1889, 761-780. Also in *Wiss. Abhandlungen*, 3, 289-332.

HELMHOLTZ, H. von (1890), Die Energie der Wogen und des Windes, *Sitz. der Akad. der Wiss.*, Berlin, 1890, 853-872. Also in *Wiss. Abhandlungen*, 3, 333-355.

HESSELBERG, T. (1926), Über Reibung und Dissipation in der Atmosphäre, *Geofys. Publ.* (Oslo), 3 (5), 1-26.

HESSELBERG, T. and H.U. SVERDRUP (1915 *a*), Beitrag zur Berechnung der Druck-und Massenverteilung im Meere, Bergens Museums Aarbok, 1914-15, (14), 1-17.

HESSELBERG, T. and H.U. SVERDRUP (1915 *b*), Die Stabilitätsverhältnisse des Seewassers bei vertikalen Verschiebungen, Bergens Museums Aarbok, 1914-15 (15), 1-16.

HOILAND, E. (1944), On the wave motions in sliding layers with internal static stability, *Archiv. Math. Naturvidensk.*, ser. B, 67, 41-72.

HOLZMAN, B. (1943), The influence of stability on evaporation, *Ann. N.Y. Acad. Sci.*, 44, 13-18.

HUTCHINSON, G.E. (1937), A contribution to the limnology of arid regions, *Trans. Connecticut Acad. Arts Sci.*, 33, 47-132.

HYLLERAS, E.A. (1939), Über die Schwingungen eines stabil geschichteten, durch Meridione begrenzten Meeres, *Astrophys. Norveg.*, 3, 139-164.

HYLLERAS, E.A. and W. ROMBERG (1941), Über die Schwingungen eines stabi 1 geschichteten, durch Meridione begrenzten Meeres, *Astrophys. Norveg.*, 3, 247-271.

JACOBS, W.C. (1937), Atmospheric waves on isentropic surfaces as evidenced by interfrontal ceiling oscillations, *Mon. Wea. Rev.*, 65, 9-12.

JACOBSEN, J.P. (1913), Bidrag til de Danske Farvandes Hydrografi, Copenhagen, 89 pp. and figs. Also available as Beitrag zur Hydrographie der dänischen Gewässer, *Komm. F. Havunders. Medd.*, Ser. Hydr. Copenhagen, 1913, 2 (2), 1-94.

JACOBSEN, J.P. (1930), Remarks on the determination of the movement of the water and the intermixing of the watersheets in a vertical direction, *Cons. perm. int. pour l'exp. de la mer*, Rapp. et Proc. verb., 64, 59-68.

JACOBSEN, J.P. & H. THOMSEN (1933), Periodical variations in temperature and salinity in the Straits of Gibraltar, pp. 275-293, ed. *R.J. Daniel, The J. Johnstone Memorial Volume*, University of Liverpool (1935).

JOHNSON, D.W. (1919), Shore Processes and Shoreline Development, *Wiley*, N.Y.

JOHNSON, D.W. (1938), Origin of Submarine Canyons, *J. Geomorph.*, N.Y., 1, 111-129, 230-243, 324-339, (1938), 2, 42-60, 133-158, 213-236, (1939).

KALLE, K. (1942), Über die innere thermische Unruhe des Meeres, *Ann. Hydrog. Mar. Meteor.*, 1942, 383-7.

KAMPE, H.J. aufm (1940), Über Schweheballon-Vermessungen in Lee der Rossittener Wanderdüne und einige allgemeine Bermerkungen über Schwingungen in der freier Atmosphäre, *Beitr. Phys. Freien Atmos.*, 26, 107-118.

KELVIN, Lord W. (1871), (W. THOMSON), The influence of frictionless wind on waves in water supposed frictionless, A letter to Prof. Tait, *Baltimore Lectures*, London (1904), App. G., 590-3.

KELVIN, Lord W. (1882), (W. THOMSON), On the thermodynamic acceleration of the earth's rotation, *Proc. Roy. Soc. Edinb.*, 11, 396-405.

KINDLE, E.M. (1927), The role of thermal stratification in lacustrine sedimentation, *Roy. Soc. Canada, Proc. and Trans.*, ser. 3, 21 (sec. 4), 1-35.

KNUDSEN, M. (1910), Danish hydrographic investigations at the Faroe Islands in the Spring of 1910, Kommis. Havundersogelser, *Ser. Hydrog.*, Copenhagen, 11 (1), 1-19.

KOCHIN, N.E. (1928), (N. KOTCHINE), Détermination rigoureuse des ondes permanentes d'ampleur finie à la surface de séparation de deux liquides de profondeur finie, *Math. Annalan*, 98, 582-615.

KOCHIN, N.E. (1938), The three dimensional problem of waves at the surface of separation of two fluids of different density caused by unevenness of the bottom, (Russian), *Trudy Glav. Geofiz. Observ.*, n.8, 3-30. Cited in *Ency. of Phys.*, 9, 489.

KOTCHINE, N. (1928), *See Kochin*, N.E. (1928).

KRÜMMEL, O. (1907), Handbuch der Ozeanographie, 2 vols. 531 and 764 pp., pub. *Engelhorn*, Stuttgart.

KUENEN, P.H. (1937), Experiments in connection with Daly's hypothesis on the formation of submarine canyons, *Leidsche Geol. Mededeelingen*, 8, 327-351.

KULLENBERG, B. (1932), A recording boundary gauge, *Goteborg's K. Vit.-o.Vitterh.-Samh. Handl.*, ser. 5B, 9 pp.

KULLENBERG, B. (1935), Internal waves in the Kattegat, *K. Svenska Hydro.-Biol Komm. Skrifter*, N.S. Hydrog., 12.

LAMB, H. (1879), Treatise on the Mathematical Theory of the Motion of Fluids, *Cambridge U. Press.*

LAMB, H. (1910), On atmospheric oscillations, *Proc. Roy. Soc. London*, ser. A, 84, 551-572.

LAMB, H. (1916), On waves due to a travelling disturbance, with an application to waves in superposed fluids, *Phil. Mag.*, ser. 6, 31, 386-398.

LAMB, H. (1932), Hydrodynamics, *Cambridge U.P.*, 6th ed. of Lamb (1879).

LAPLACE, P.S. Marquis de (1807), Mécanique Céleste, Livre 4 Des Oscillations de la Mer et de l'Atmosphère, *Crapelet*, Paris, ch. 5, 310-4, Also in Œuvres complètes, *Acad. Sci. Paris, Gauthier-Villars*, Paris (1878), vol. 2.

LEK, L. (1938 *a*), Die Ergebnisse der Strom-und Serienmessungen, *Sci. Res. Snellius Exped.* 1929-1930, 2 (3), 1-52.

LEK, L. (1938 *b*),Interne Wellen in den niederlandischostindischen Gewässern, *15th Int. Geogr. Cong.*, Amsterdam 12 (sec. 2 *b*), 69-76.

LEK, L. and J.E. FJELDSTAD (1938), Interne Wellen auf Ankerstation 253 *a*, Sci. Res. Snellius Exped. 1929-1930, *E.J. Brill*, Leichen, 2, 152-169.

LOVE, A.E.H. (1891), Wave-motion in a heterogeneous heavy liquid, *Proc. London Math. Soc.*, ser. 1, 22, 307-316.

LYRA, G. (1940), Über den Einfluss von Bodenerhebungen auf die Strömung einer stabil geschichteten Atmosphäre, *Beitr. Phys. Freien Atmos.*, 26, 197-206.

LYRA, G. (1943), Theorie der stationären Leewellenströmung in freier Atmosphäre, *Z. Ang. Math. Mech.*, 23, 1-28.

MAKKAWEEV, W.M. (1936), Bemerkungen über die Theorie der langen internen Wellen, *Verein. Hydrolog. Konf. Baltischen Staaten*, Helsinki (1936), Ber. 15C, 13-19.

MANLEY, G. (1945), The Helm Wind of Crossfell, *Quart. J. Roy. Met. Soc.*, 71, 197-219.

MEYER, H. (1904), Totwasser, *Ann. Hydrog. Mar. Meteor.*, 32, 20-28.

MONISH, B.H. (1938), Discussion of Grover and Howard (1938), *Trans. Amer. Soc. Civil Eng.*, 103, 751-5.

MONTGOMERY, R.B. (1938), Circulation in the upper layers of the southern North Atlantic with the use of isentropic analysis, *Pap. Phys. Oceanog. Meteorol.*, 6 (2), 1-55.

MORTIMER, C.H. (1941), The exchange of dissolved substances between mud and water in lakes, *J. Ecol.*, 29, 280-329.

MORTIMER, C.H. (1942), The exchange of dissolved substances between mud and water in lakes, *J. Ecol.*, 30, 147-201.

MUNK, W.H. (1941), Internal waves in the Gulf of California, *J. Marine Research*, 4, 81-91.

NANSEN, F. (1900), The Oceanography of the North Polar Basin, Sci. Res. Norwegian Polar Exped. 1893-6, *Longmans Green & Co.*, London, 3, 1-427.

NEUMAN, F. von (*c* 1835), Einleitung in die Theoretisch Physik, *Teubner*, Leipzig, 1883, p. 146.

PARR, A.E. (1936), On the probable relationship between vertical stability and lateral mixing processes, *J. Internat. Assoc. Explor. Sea*, 11 (3), 308-313.

PEKERIS, C.L. (1937), Atmospheric oscillations, *Proc. Roy. Soc. London*, ser. A, 158, 650-671.

PETTERSSON, H. (1920), Internal movements in coastal waters and meteorological phenomena, *Geogr. Ann.*, 2, 33-66.

PETTERSSON, H. (1931), Eddy-viscosity in stratified water, *K. Vet.-o. Vitterh.-Samh. Handl.*, Göteborg, ser. 5 B, 2 (1), 1-21.

PETTERSSON, H. (1938), Internal waves in Scandinavian waters, *15th Int. Geogr. Congr.*, Amsterdam, 12 (sec 2 *b*), 77-86.

PETTERSSON, H. and B. KULLENBERG (1933), Boundary waves in the Kattegat, *K. Vet.-o. Vitterh.-Samh. Handl.*, Göteborg, ser. 5 B, 3 (9), 1-13.

PETTERSSON, H. and H.W. MALMHEDEN (1935), Eddy-viscosity in stratified water, II, *K. Vet.-o. Vitterh.-Samh. Handl.*, Göteborg, ser. 5 B, 4 (8), 1-14.

PETTERSSON, O. (1894), A review of Swedish hydrographic research in the Baltic and North Seas, *Scott. Geogr. Mag.*, 10, 281-302, 352-9, 413-427, 449-462, 525-539, 617-635, maps.

PETTERSSON, O. (1909), Gezeitähnliche Bewegungen des Tiefen wassers, Perm. Int. Assoc. Expl. Mer. Publ. de Circonstance, n° 47, 1-21.

PETTERSSON, O. (1913), Climatic variations in historic and prehistoric time, *Svenska Hydro.-Biol. Komm. Skrifter*, 5, 1-26.

PETTERSSON, O. (1923), Innere Bewegungen in den Zwischenschicht des Meeres und der Atmosphäre, *Nova Acta Regiae Soc. Sci. Upsaliensis*, ser. 4, 6, 1-43.

PETTERSSON, O. (1933), Interne Gezeiten-Wellen, Rapp. Proc. Verb. Int. Council Expl. Sea, 82, 1-26, I-VIII.

PETTERSSON, O. (1934), Ebb och flod i atmosfären (Swedish with French resume), *K. Vet.-o Vitterh. -Samh. Handl.*, Göteborg, ser. 5 B, 3 (9), 1-13.

PETTERSSON, O. (1935), Klimatets Rörlighet (first part in Swedish, second in English), *K. Vet.-o. Vitterh.-Samh. Handl.*, Göteborg, ser. 5 B, 4 (6), 1-105.

PETTERSSON, O. and G. EKMAN (1891), Grunddragen af Skageraks och Kattegats Hydrografi, *K. Svenska Vet. Akad. Handl.*, 24 (11), 1-162 and figs.

PLINIUS SECUNDUS, C. (c. 77), Historia Naturalis, Book 2, (An account of the World and its elements). Ch. 106; transl. J. Bostock and H.T. Riley, *pub. H.G. Bohn.* London (1855).

POCKELS, F. (1913), Zur theoretische Berechnung der Temperaturverteilung in der freien Atmosphäre bei Föhn, *Meteorol. Z.*, 30, 216-222.

PRANDTL, L. (1929), Einfluss stabilisierender Kräfte auf die Turbulenz, Vortrage a.d. Gebiete der Aero. u.verwandter Gebiete, Aachen, ed. *Gilles, Hoff, von Karman;* pub. *Springer,* Berlin pp. 1-7. Also in Gesamm. Abhandl., éd. *Tollmien, Schlichting and Görtler,* pub. *Springer,* Berlin (1961), 2, 7718-7853, transl. by J. Vanier, N.A.C.A. TM 625, Washington (1931), 11 pp.

PRANDTL, L. (1932), Meteorologische Anwendung der Strömungslehre, Beitr. Phys. Freien Atmos., 19, 188-202. Also in Gesamm. Abhandl (see above), 3, 1081-1099.

PRANDTL, L. (1933), Correction to Prandtl (1930) q.v., published with it in Gessammelte Abhandlungen.

PRANDTL, L. (1944 *a*), Zur Frage des vertikalen Turbulenz - Wärmestromes, *Meteorol. Z.*, 61, 12-14. Also in Gesamm. Abhandl. (see above), 3, 1157-1162.

PRANDTL, L. (1944 *b*), Nochmals der vertikale Turbulenz - Wärmestrom, *Meteorol. Z.*, 61, 169-170. Also in Gesamm. Abhandl. (see above), 3, 1163-6.

PRANDTL, L. (1949 *b*), Führer duch die Strömungslehre, *Springer-Verlag,* Berlin. Trans. as "Essentials of Fluid Dynamics", *Blackie & Sons,* London (1952), 452 pp.

PRANDTL, L. & H. REICHARDT (1934), Einfluss von Wärmeschichtung auf die Eigenschaften einer turbulenten Strömung, Deutsche Forschung, aus der Arbeit der Notgemeinschaft der Deutschen Wiss., 21, 110-121. Also in Gesamm. Abhandl (see above), 2, 846-855.

PRIESTLEY, H.J. (1909), On the oscillations of superposed fluids, *Proc. Cambr. Phil. Soc.*, 15, 297-309.

QUENEY, P. (1936), Recherches relatives à l'influence du relief sur les éléments météorologiques, *La Météorologie,* 79, 335-353, 453-470.

QUENEY, P. (1941), Ondes de gravité produltes dans un courant aérien par une petite chaîne de montagnes, *Compt. Rend. Acad. Sci. Paris,* 213, 588-591.

RAETHJEN, P. (1929), Zur Vertikalbewegung im atmosphärischen Kontinuum III Teil, *Meteorol. Z.*, 46, 383-393; IV Teil, 46, 420-435.

RAETHJEN, P. (1932), Die Luftunruhe in der freien Atmosphäre und ihre Beobachtungen in Flugzeug, *Meteorol. Z.*, 69, 62-67.

RAYLEIGH, (Lord) (1883), Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, *Proc. London Math. Soc.*, 14, 170-7. Also in *Sci. Papers,* 2 (100), 200-7.

REVELLE, R.R. (1939), Sediments of the Gulf of California, *Bull. Geol. Soc. Amer.*, 50, 1929.

REYNOLDS, O. (1883), An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and the law of resistance in parallel channels, *Proc. Roy. Soc. London,* ser. A, 35, 84-99.

RICHARDSON, L.F. (1920), The supply of energy from and to atmospheric eddies, *Proc. Roy. Soc. London,* ser. A, 97, 354-373.

RICHARDSON, L.F. (1925), Turbulence and vertical temperature difference near trees, *Phil. Mag.*, ser. 6, 49, 81-90.

ROSENHEAD, L. (1932), The formation of vortices from a surface of discontinuity, *Proc. Roy. Soc. London,* ser. A, 134, 170-192.

ROSSBY, C.G. and R.B. MONTGOMERY (1935), The layer of frictional influence in wind and ocean currents, *Pap. Phys. Oceanogr. Meteorol.*, 3, 1-101.

RUBEY, W.W. (1938), disc. of The passage of turbid water through Lake Mead, by N.C. Grover and C.S. Howard, *Trans. Amer. Soc. Civil Eng.*, 103, 720-732, disc. by Rubey, 103, 763-7.

SANDSTRÖM, J.W. (1908), Dynamische Versuche mit Meerewasser, *Ann. Hydrog. Mar. Meteor.*, 1908, 6-23.

SCHLICHTING, H. (1935), Turbulenz bei Wärmeschichtung, *Z. Ang. Math. Mech.*, 15, 313-338. Transl. as N.A.C.A. TM 1262, Washington, 1950, 55 pp. (note that trans. is not completely reliable).

SCHMIDT, J. (1922), Oceanography of the Gibraltar region, *Nature,* 59, 45.

SCHMIDT, J. (1929), Introduction to the oceanographical reports, *The Danish "Dana" Exped.*, 1920-2, Copenhagen, N.1, 69-70.

SCHMIDT, W. (1908), Stehende Schwingungen in der Grenzschicht zweier Flüssigkeiten, *Sitzungsb. Akad. Wiss. Wien,* ser. II *a,* 117, 91, 102.

SCHMIDT, W. (1918), Über Arbeitsleistung und Arbeitsverbrauch in der freien Luft, *Ann. Hydrog. Mar. Meteorol.* (1918), 324-332.

SCHOTT, G. (1928), Die Wasserbewegungen im Gebiete der Gibraltarstrasse, *J. Int. Explor. Mer.*, Copenhagen, 3, 139.

SEIWELL, H.R. (1937), Short period vertical oscillations in the western basin of the North Atlantic, *Pap. Phys. Oceanogr. Meteorol.*, 5 (2), 1-44.

SEIWELL, H.R. (1938), Über die ganz- und halbtägige Mondkomponente der internen Wellen in westlichen Nordatlantischen Ozean, *Ann. Hydrog. Marit. Meteor.*, 66, 485-7.

SEIWELL, H.R. (1939), The effect of short period variations of temperature and salinity on calculations in dynamic oceanography, *Pap. Phys. Oceanog. Meteorol.*, 7 (3), 1-32.

SEIWELL, H.R. (1941), Summary of some results of internal wave investigations in the North Atlantic, *Trans. Amer. Geophys. Union,* 22, 494-8.

SEIWELL, H.R. (1942), An analysis of vertical oscillations in the southern North Atlantic, *Proc. Amer. Philos. Soc.*, 85, 136-158.

SOLBERG, H. (1928), Integrationen der atmosphärischen Storungsgleichungen, *Geofys. Publ.*, Oslo, 5 (9), 120 pp.

SOLBERG, H. (1936), Schwingungen und Wellenbewegungen in einer Atmosphäre mit nach oben abnehmender Temperatur, *Astrophs. Norveg.*, 2, 123-172.

STOKES, G.C. (1847), On the theory of oscillatory waves, Trans. Cambridge Phil. Soc., 8, 441. Also in *Math. Phys. Pap.*, pub. 1880, 1, 212-219.

STORMER, C. (1939), Astrophys. Norveg., 3 (5), 117-138. Abstract in A notable meteor, *Nature,* 1940, 145, 195.

STÜMKE, H. (1940), Beitr. Phys. Freien. Atmos.. 26, 207, cited by Prandtl (1949).

SUDA, K. (1932), On the dissipations of energy in the density current, *Geophys. Mag.* (Tokyo), 6, -314 and figs.

SUDA, K. (1936), On the dissipations of energy in the density current, *Geophys. Mag.* (Tokyo), 10, 131-243.

SVERDRUP, H.U. (1936), The eddy conductivity of the air over a smooth snow field, *Geophys. Publ.* (Oslo). 9 (7), 1-69.

SVERDRUP, H.U. (1938), On the influence of stability and instability on the wind profile and eddy conductivity near the ground, Proc. 5th Int. Cong. App. Mech. 1938, *Wiley,* N.Y. (1939), pp. 369-372.

SVERDRUP, H.U. (1939), Lateral mixing in the deep water of the South Atlantic Ocean, *J. Marine Research,* 2, 195-207.

SVERDRUP, H.U. (1940), The Gulf of California, *Assoc. Oceanog. Phys. Proc.-Verb.*. No. 3, 170-1.

SVERDRUP, H.U., M.W. JOHNSON and R.H. FLEMING (1942), The Oceans, *Prentice Hall,* N.Y., 1 080 pp.

SYNGE, J.L. (1933), The stability of heterogeneous liquids, *Trans. Roy. Soc. Canada,* 27 (pt. 3), 1-18.

TAKEGAMI, T. (1936), A study of the effect of a local wind upon the sea-surface and on the development of the internal boundary wave, *Mem. Kyoto Imperial Univ., College of Science.* ser. A, 19, 109-130.

TAYLOR, G.I. (1914), Adams prize essay of 1915, pub. as Effect of variation of density on the stability of superposed streams of fluids, *Proc. Roy. Soc. London,* 1931, ser. A, 132, 499-523. Also in Sci. Pap., ed. *G.K. Batchelor, Cambridge U. Press,* 1960, 2. 219-239.

TAYLOR, G.I. (1927 *a*), An experiment on the stability of superposed streams of fluid, *Proc. Cambridge Phil. Soc.*, 23, 730-1. Also in Sci. Papers, ed. *G.K. Batchelor, Cambridge U. Press,* 1960, 2, 193-4.

TAYLOR, G.I. (1927 *b*), Turbulence, *Quart. J. Roy. Met. Soc.*, 53 pp., 201-211.

TAYLOR, G.I. (1929), Waves and tides in the atmosphere, *Proc. Roy. Soc. London*, ser. A, 126, 169-183. Also in Sci. Papers, ed. *G.K. Batchelor, Cambridge U. Press*, 1960, 2, 201-212.

TAYLOR. G.I. (1931 *a*), - see Taylor (1914).

TAYLOR, G.I. (1931 *b*), Internal waves and turbulence in a field of variable density, *Cons. perm. int. pour l'explor. de la mer*, Rapp. et proc. verb. 76. Also in Sci. Papers, ed. *G.K. Batchelor, Cambridge U. Press*, 1960, 2, 240-6.

TAYLOR, G.I. (1932), The resonance theory of semidiurnal atmospheric oscillations, *Mem. Roy. Met. Soc.*, 4, 43-51. Also in Sci. Papers, ed. *G.K. Batchelor. Cambridge U. Press*, 1960, 2, 271-9.

TAYLOR, G.I. (1936). The oscillations of the atmosphere, *Proc. Roy. Soc. London*, ser. A, 156, 318-326. Also in Sci. Papers, ed. *G.K. Batchelor, Cambridge U. Press*, 1960, 2, 365-371.

THORNTHWAITE, C.W. and P. KASER (1943), Wind-gradient observations, *Trans. Amer. Geophys. Union*, 24, 166-182.

THOULET, J. (1894), Contribution à l'étude des lacs des Vosges, *Géographie*, 15. Cited by Krümmel (1907) and others.

VÄISÄLÄ, V. (1925), Über die Wirkung der Windschwankungen auf die Pilotbeobachtungen, *Soc. Sci. Fennica Comm. Phys.-Math.*, 12 (19), 1-46.

WATSON, E.R. (1903), Internal oscillations in the waters of Loch Ness, *Nature*, 69, 174.

WATSON, E.R. (1904), Loch Ness, II Movements of the waters of Loch Ness as indicated by temperature observations, *Geogr. J.* (London), 24, 430-7.

WEBB, R.R. (1884), Math. Tripos. Exam., Cambridge (Jan. 1884), cited by Greenhill (1887).

WEDDERBURN, E.M. (1907), An experimental investigation of the temperature changes occuring in fresh water lochs, *Proc. Roy. Soc. Edinb.*, 28, 2-20.

WEDDERBURN, E.M. (1908), Temperature observations in Loch Garry, *Proc. Roy. Soc. Edinb.*, 29, 98-135.

WEDDERBURN, E.M. (1909), Dr. O. Pettersson's observations on deep water oscillations, *Proc. Roy. Soc. Edinb.*, 29, 602-6.

WEDDERBURN, E.M. (1911), The temperature seiche, Pt. 1, Temperature observations in the Madüsee, Pomerania, *Trans. Roy. Soc Edinb.*, 47, 619-636.

WEDDERBURN, E.M. (1912), Temperature observations in Loch Earn with a further contribution to the hydrodynamical theory of the temperature seiche, *Trans. Roy. Soc. Edinb.*, 48, 629-695.

WEDDERBURN, E.M. and A.M. WILLIAMS (1911), The temperature seiche. Pt. 4, Experimental verification of the temperature seiche theory, *Trans. Roy. Soc. Edinb.*, 47, 636-642.

WEDDERBURN, E.M. & A.W. YOUNG (1915), Temperature observations in Loch Earn II, *Trans. Roy. Soc. Edinb.*, 50, 741-767 and pl.

WEGENER, A. (1906), Studien über Luftwogen, *Beitr. Phys. freien Atmos.*, 2, 55-72.

WEGENER, A. (1912), Nachtrag zu den "Studien über Luftwogen". *Beitr. Phys. freien Atmos.*, 4, 23-25.

WHEELER, W.H. (1906), A Practical Manual of Tides and Waves, *Longmans Green*, London, pp. 96, 102.

ZEILON, N. (1913), On the seiches in Gullmar Fiord, *Svenska Hydro.-Biol. Komm Skrifter*, v. 5, 1-18.

ZEILON, N. (1915). On the influence of diffusion upon the propagation of boundary waves, *Nov. Acta Reg. Soc. Sci. Upsaliensis*, ser. 4, 4 (2), 1-16.

ZEILON, N. (1917), On tidal boundary waves and related hydrodynamical problems, *K. Svenska Vet. -Akad. Handl.*, 47 (4), 1-49.

ZEILON, N. (1934), Experiments on boundary tides, *K. Vet.-o. Vitterh. -Samh. Handl.*, Göteborg, ser. 5 B, 3 (10). 1-12.

ZÖPPRITZ, K. (1878), Hydrodynamische Probleme in Beziehung zur Theorie der Meeresströmungen, *Wiedemanns Ann. Phys. Chemie*, Leipzig, N.F., 3, 582-607.

OTHER REFERENCES

CHAPMAN, S. (1951), Atmospheric tides and oscillations, Compendium of Meteorology, ed. *T.F. Malone, Amer. Meteorol. Soc. Boston*, pp. 510-530.

DEACON. E.L. (1949), Vertical diffusion in the lowest layers of the atmosphere, *Quart. J. Roy. Met. Soc.*, 75, 89-103.

DEFANT, A. (1961), Physical Oceanography, *Pergamon*, 2 vols; In particular Chapter 16, vol. 2, Internal Waves.

HINWOOD. J.B. (1970), The study of density-stratified flows up to 1945, Part I, *La Houille Blanche*, 1970, 347-359.

HUTCHINSON, G.E. (1957), A Treatise on Limnology. *Wiley*, N.Y., vol. 1, 1 015 pp.

KATO, H. and O.M. PHILIPS (1969), On the penetration of a turbulent layer into a stratified fluid, *J. Fluid Mech.*, 37, 643-655.

KEULEGAN, G.H. (1949), Interfacial instability and mixing in stratified flows, *J. Research. U.S. Nat. Bureau Standards*, RP 2040.

LAFOND, E.C. and C.S. COX (1962), Internal waves, Ch. 22 of The Sea, ed. *M.N. Hill*, Interscience, N.Y., 1. 731-763.

LONG, R.R. (1953), Some aspects of the flow of stratified fluids I, *Tellus*, 5, 42-58.

LONG, R.R. (1954), Some aspects of the flow of stratified fluids II, *Tellus*, 6, 97-115.

LONG, R.R. (1955). Some aspects of the flow of stratified fluids III, *Tellus*, 7, 347-357.

MACAGNO, E.O. and H. ROUSE (1961), Interfacial mixing in a stratified flow, *Proc. Amer. Soc. Civil Engin.*, 87 (EM 5), 55-81.

MALONE, T.F. (1951), ed. *Compendium of Meteorology, Amer. Meteorol. Soc. Boston*, 1 334 pp.

MILES, J.W. (1960), On the generation of surface waves by turbulent shear flows, *J. Fluid Mech.*. 7, 469-478.

MILES, J.W. (1967), Internal waves in a continuously stratified atmosphere or ocean, *J. Fluid. Mech.*, 28, 305-310.

MORTIMER, C.H. (1953), The resonant response of stratified lakes to wind, *Schweitz. Z. Hydrol.*, 15, 94-151.

NEUMANN, G and W.J. PIERSON Jr. (1966), Principles of Physical Oceanography, *Prentice Hall*, N.Y.. 545 pp.

PRANDTL, L. (1949), Führer durch die Strömungslehre, *Springer*, Berlin; transl. as Essentials of Fluid Dynamics, *Blackie & Sons*, London (1952), 452 pp.

PRIESTLEY, C.H.B. and W.C. SWINBANK (1947), Vertical transport of heat by turbulence In the atmosphere, *Proc. Roy. Soc. London*, ser. A, 189, 543-561.

PRITCHARD. D.W. (1956), The dynamic structure of a coastal plain estuary, *J. Marine Research*, 15, 33-42.

PROUDMAN, J. (1953), Dynamical Oceanography, *Methuen*, London, 409 pp.

QUENEY, P. (1947), Theory of perturbations in stratified currents with applications to air flow over mountain barriers, Univ. of Chicago, Dept. Meteorol. Misc. Rept. 23, 81 pp.

QUENEY, P. (1948), The problem of air flow over mountains : a summary of theoretical studies. *Bull. Amer. Meteorol. Soc.*, 29, 16-26.

QUENEY, P., G.A. CORBY, N. GERBIER, H. KOSCHMIEDER and J. ZIEREP (1960), The Airflow over Mountains, *World Meteorol. Organisation*, Tech. Note 34, World Meteorol. Org., Geneva, 135 pp.

ROSSBY, C.G. (1951), On the vertical and horizontal concentration of momentum in air and ocean currents Pt. I, *Tellus*, 3, 15-27.

ROUSE, H. & J. DODU (1955), Diffusion turbulente à travers une discontinuité de densité, *La Houille Blanche*, 1955, 522-532.

SQUIRE, H.B. (1933), On the stability of the three-dimensional disturbances of viscous flow between parallel walls, *Proc. Roy. Soc. London*, ser. A, 142, 621-8.

STUART, J.T. (1960), Non-linear effects in hydrodynamic stability, *Proc. 10th Internat. Congr. App. Mech.*, 63-97.

YIH, C.S. (1965), The Dynamics of Non-Homogeneous Fluids, *Macmillan*, N.Y., 306 pp.