ON FLOW IN ESTUARIES

PART III

Derivation of general and breadth integrated models

by

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ON FLOW IN ESTUARIES

PART III: DERIVATION OF GENERAL AND BREADTH INTEGRATED MODELS

I. INTRODUCTION.

The general mathematical model of the flow in estuaries leads to very complicated nonlinear partial differential equations which at the present stage cannot be solved even if a satisfactory way of treating the turbulent diffusion existed. It is therefore necessary to derive simpler models which we can solve or at least from which we can obtain some quantitative or qualitative information. In order for such simplified models to be of value it is very important that the assumptions made at each stage of the derivation are clearly understood so that one can judge the applicability of each model to a particular estuarine flow.

There are two basic methods for simplifying the full mathematical model. In the first we use order of magnitude arguments to show that certain terms are very much smaller than others and hence can be neglected. In applying this method one must be careful not to differentiate the resulting equations since a term may be small while its derivative is not. The second method consists of reducing the dimensionality of the model from three to two or even one dimension. In order to do this one argues either that the velocity component in one direction is zero or, preferable, that the variations of all the variables in the

direction are so small that they can be ignored. Both these methods of simplifying the equations can be applied simultaneously.

Some of the models discussed in this paper have been derived for slightly stratified estuaries by Pritchard (1), (2) and Rattray and Hansen (3), (4), (5). This work has been discussed in detail in the first paper in this series, Rasmussen and Hinwood (6) which will be referred to as Part I. Some limitations of these models have been derived in Rasmussen and Hinwood (7), referred to as Part II.

The basic mathematical formulation of the problem is given in Section 2. Some fairly general three-dimensional models are derived in Section 3 where we first take the curl of the full dynamic equations before any simplification is carried out so that no gross errors are introduced by the derivative of a neglected term being large. In Section 4 the full equations are integrated over the breadth of the estuary and a two-dimensional model is derived. Boundary conditions are introduced as required; a comprehensive discussion of boundary conditions will be deferred to a later paper. In Section 5 we discuss the different models obtained in the paper and indicate the directions in which more work is required. One obvious extension is to obtain depth intequated models applicable to wide shallow bays and also one-dimensional models, and these will form the topics of later papers in this series.

It should be noted that the treatment of the turbulence terms is a separate problem to the simplification of the model by elimination of terms and reduction of dimensionality considered here. The turbulence terms are left in their basic form where possible, and only the conventional diffusion coefficient substitution is proposed.

2. Formulation

Let the origin of the cartesian coordinate system be located in the free surface at the upstream limit of the estuary, and let x be the longitudinal coordinate taken positive seawards, y the lateral coordinate, and z the vertical coordinate positive downwards. If $\underset{\sim}{u} = (u_x, u_y, u_z)$ and $\underset{\sim}{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ are the velocity vector and the vector of the earth's rotation and p and ρ the pressure and density, the equations of motion can be written in the form

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = -\nabla p + \rho \underset{\sim}{u} \times \underset{\sim}{\Omega} + g\rho\hat{k} \qquad (2.1)$$

where $\hat{k}$ is the unit vector in the z direction.

The viscous stress terms are not explicitly included in these quations since they are much smaller than the turbulent stress terms and may be regarded as being included in the Reynolds stresses. The continuity equation for the instantaneous velocity $\underset{\sim}{u}$ is

$$\nabla\cdot\underset{\sim}{u} = 0 \qquad (2.2)$$

where the water is regarded as incompressible.

If S(x,y,z,t) is the salinity, the steady state equation

for salt conversation is

$$\frac{\partial S}{\partial t} + \nabla\cdot(S\underset{\sim}{u}) = 0 \qquad (2.3)$$

where molecular diffusion is neglected on grounds that the turbulent diffusion is larger by several orders of magnitude. The density ρ and S are related by an equation of state

$$\rho = \rho_o(1 + kS) \qquad (2.4)$$

where ρ_o is a reference density, e.g. that of fresh water and k a constant, $k \ll 1$.

The length of the estuary is denoted by L and D is taken to be a characteristic depth. For most estuaries $L \gg D$.

3. General Models

We shall now show how the governing equations (2.1) to (2.4) can be simplified systematically to produce approximate models which are easier to analyse. Pressure is eliminated by taking the curl of the equation of motion before averaging, then by consideration of orders of magnitude, the smaller coriolis and inertial terms are dropped. In this section we make no assumption about the extent of the stratification.

3.1 The Vorticity Equation

If we take the curl of equation (2.1) we obtain

$$\nabla \times (\rho \frac{\partial \underset{\sim}{u}}{\partial t}) + \nabla\rho \, [(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u}] + \rho\nabla \times [(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u}]$$

$$= \nabla\rho \times (\underset{\sim}{u} \times \underset{\sim}{\Omega}) + \rho\nabla \times (\underset{\sim}{u} \times \underset{\sim}{\Omega}) + g\nabla\rho \times \hat{k} \, .$$

Since $\nabla \times (\underset{\sim}{u} \times \underset{\sim}{\Omega}) = (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{\Omega} + (\underset{\sim}{\Omega}\cdot\nabla)\underset{\sim}{u} + \underset{\sim}{u}(\nabla\cdot\underset{\sim}{\Omega})$ and since the length scale for variation in $\underset{\sim}{\Omega}$ is very much greater than that for variation in $\underset{\sim}{u}$ the term $\underset{\sim}{u}(\nabla\cdot\underset{\sim}{\Omega}) - (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{\Omega}$ may be neglected in comparison with the term $(\underset{\sim}{\Omega}\cdot\nabla)\underset{\sim}{u}$. Furthermore $\nabla\rho = \rho_o k\nabla S$ and since $k \ll 1$, we can neglect the terms containing $\nabla\rho$ in comparison with those containing ρ. Hence we have

$$\nabla \times \left[(1 + kS) \frac{\partial \underset{\sim}{u}}{\partial t}\right] + \nabla \times [(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}] = (\underset{\sim}{\Omega}\cdot\nabla)\underset{\sim}{u} + \hat{j}\, kg \frac{\partial S}{\partial x} + \hat{i}\, kg \frac{\partial S}{\partial y} \qquad (3.1)$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors in the x and y directions, respectively.

The velocity vector $\underset{\sim}{u}$ may be expressed in the form

$$\underset{\sim}{u} = \bar{\underset{\sim}{u}} + \underset{\sim}{U} + \underset{\sim}{u}' \qquad (3.2)$$

where $\bar{\underset{\sim}{u}}$ = the time mean velocity averaged over one or more tidal cycles,

$\underset{\sim}{U}$ = the tidal velocity,

$\underset{\sim}{u}'$= the turbulent velocity fluctuation which is assumed to have a time scale significantly smaller than that of the tidal period.

These constituent velocities each satisfy the continuity equation

$$\nabla\cdot\bar{\underset{\sim}{u}} = 0 \qquad (3.3a)$$

$$\nabla\cdot\underset{\sim}{U} = 0 \qquad (3.3b)$$

$$\nabla\cdot\underset{\sim}{u}' = 0. \qquad (3.3c)$$

We now obtain the time mean form of the components of (3.1)

by substituting (3.2) and taking the time mean of the resultant equations. In the operation of taking the time mean, terms of the type $\overline{\bar{u}_x u'_y}$ and $\overline{\bar{u}_x U_y}$ are set equal to zero since by the Reynolds axioms $\overline{\bar{u}_x u'_y} = \bar{u}_x\overline{u'_y} = 0$ and similarly $\overline{\bar{u}_x U_y} = \bar{u}_x \overline{U_y} = 0$. It is assumed that terms of the type $\overline{U_x u'_y}$ are equal to zero since there is no reason to suspect a correlation between the oscillating tidal motion and the turbulent velocity fluctuations provided that there is a gap in the velocity spectrum between the turbulence and the tidal motion (Charnock, 8). In order that the operation of taking the time mean can be carried out, it is necessary that the change of the crossectional area of the estuary as the tide rises and falls is fairly small. This will be supposes to hold for the class of estuaries that are considered in this paper.

By considerations of orders of magnitude, the least important terms in the vorticity equation (3.1) will now be eliminated treating each component in turn. The component in the y direction is

$$\frac{\partial}{\partial z}\{(1 + kS)\frac{\partial u_x}{\partial t}\} - \frac{\partial}{\partial x}\{(1 + kS)\frac{\partial u_z}{\partial t}\} + \frac{\partial}{\partial z}(\underset{\sim}{u}\cdot\nabla)u_x - \frac{\partial}{\partial x}(\underset{\sim}{u}\cdot\nabla)u_z =$$

$$(\underset{\sim}{\Omega}\cdot\nabla)u_y + kg\frac{\partial S}{\partial x} \; . \qquad (3.4)$$

The convective terms may be written as

$$C_y = \frac{\partial}{\partial z}(\frac{\partial}{\partial x}u_x u_x + \frac{\partial}{\partial y}u_x u_y + \frac{\partial}{\partial z}u_x u_z) - \frac{\partial}{\partial x}(\frac{\partial}{\partial x}u_x u_z + \frac{\partial}{\partial y}u_y u_z + \frac{\partial}{\partial z}u_z u_z)$$

where the continuity equation (2.2) has been used. Denote the orders of magnitude of u_x, u_y, u_z by U_1, U_2, U_3 and those of x, y, z by γ, b_1, h. A conservative estimate is that $U_2 = U_3 = 0.1U_1$ and that $h = \gamma/10$. Then the ratio of each term of the first group to those of the second is in order of 10^2, so that C_y may be approximated by C_y^* where

$$C_y^* = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} u_x u_x + \frac{\partial}{\partial y} u_x u_y + \frac{\partial}{\partial z} u_x u_z\right).$$

If (3.2) is substituted into this expression for C_y^* and the time mean taken, then by use of the continuity equation (3.3) the following expression is obtained

$$\begin{aligned} \overline{C_y^*} = & \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \bar{u}_x\bar{u}_x + \frac{\partial}{\partial y} \bar{u}_x\bar{u}_y + \frac{\partial}{\partial z} \bar{u}_x\bar{u}_z\right) \\ & + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \overline{U_xU_x} + \frac{\partial}{\partial y} \overline{U_xU_y} + \frac{\partial}{\partial z} \overline{U_xU_z}\right) \\ & + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \overline{u'_xu'_x} + \frac{\partial}{\partial y} \overline{u'_xu'_y} + \frac{\partial}{\partial z} \overline{u'_xu'_z}\right) . \end{aligned} \qquad (3.5a)$$

Further reduction of this expression may be possible in any particular situation by considering orders of magnitude of the different terms. For example the first two turbulence terms will frequently be negligible compared with the third.

3.2 Local Accelarations

By use of equation (2.4), the local acceleration terms in equation (3.4) may be written as

$$\frac{\partial}{\partial z}\left((1 + kS) \frac{\partial u_x}{\partial t}\right) - \frac{\partial}{\partial x}\left((1 + kS) \frac{\partial u_z}{\partial t}\right) =$$

$$\left(k \frac{\partial S}{\partial z} \frac{\partial u_x}{\partial t} + (1 + kS) \frac{\partial^2 u_x}{\partial z \partial t} - k \frac{\partial S}{\partial x} \frac{\partial u_z}{\partial t} - (1 + kS) \frac{\partial^2 u_z}{\partial x \partial t}\right).$$

The salinity may now be expressed in the form

$$S = \bar{S} + S_T + S'$$

where $\bar{S}$ = the time mean salinity,

S_T = the periodic variation in salinity due to the tidal motion,

S' = the turbulent fluctuation in the salinity.

Substiting this expression and the similar one for $\underset{\sim}{u}$ and averaging over one or more tide cycles fourteen terms are obtained. The orders of magnitude considered above show that the leading turbulence, tidal and mean terms are of order

$$kS_oU_1^2/10hL, \quad kS_oU_{1T}^2/Ln \quad \text{and} \quad \bar{U}_1/h\tau$$

respectively. In obtaining these expressions it was assumed $S' = O\left(\frac{S_o\gamma}{L}\right)$, $S_T = O\left(\frac{S_oU_{1T}T}{2L}\right)$ where S_o is the salinity of sea water, the time scale for the turbulence is $T' = \gamma/U_1$, for the tidal motion it is $\frac{1}{2}T$, half the tidal period, and for the mean motion is τ, and U_1, $\bar{U}_1$ and U_{1T} are the orders of magnitude of u_x, $\bar{u}_x$ and U_x respectively; S' is likely to be over estimated. Throughout this paper we use the notation O( ) to indicate the order of magnitude of the term appearing inside the bracket.

If the time mean velocity field is changing significantly with time, i.e. if τ is small (although longer than T), the only significant term will be the mean term

$$\frac{\partial^2 \bar{u}_x}{\partial z \partial t} \, .$$

If the time-mean velocity field is steady, as will be assumed from this point onwards, the leading tidal terms are the largest of the local accelerations, but are smaller than the convective terms by the factor kS_o. Hence they may be omitted.

A simpler argument is to apply the Boussinesq approximation to eliminate the terms in kS, then average, then impose steady flow eliminating all terms. The procedure followed here has the advantage of showing which terms are being compared and of indicating their magnitudes so that the reliability of the procedure may be assessed and if necessary the leading terms may be reinstated or evaluated as a check on consistancy.

3.3 <u>Coriolis Terms</u>

Now consider the time mean coriolis terms in equation (3.4). Let Ω be the order of magnitude of Ω_x, Ω_y and Ω_z, and $\bar{U}_2$ that of $\bar{u}_y$.

Then

$$O(\Omega_x \frac{\partial \bar{u}_y}{\partial x}, \Omega_y \frac{\partial \bar{u}_y}{\partial y}, \Omega_z \frac{\partial \bar{u}_y}{\partial z}) = (\frac{\Omega \bar{U}_2}{\gamma}, \frac{\Omega \bar{U}_2}{b}, \frac{\Omega \bar{U}_2}{h}) \, .$$

In the vast majority of estuaries $\gamma >> b_1 >> h$; hence the dominant term is likely to be $\Omega_z \frac{\partial \bar{u}_y}{\partial z}$. Substitution of typical values for this term shows it to be smaller than the salinity gradient term although not necessarily smaller than the time-mean convective accelerations. It may be concluded that the coriolis terms can usually be neglected in equation (3.4) This result is supported by the experiments of Valembois and Bonnefille (9) who found that the coriolis forces produced in a model of a triangular bay mounted on a rotating table had a negligible effect on tidal amplitudes if the dimensions of the bay were less than 150 km. From their numerous current ellipses it appears that no mean currents were set up by the interactions of coriolis force, tides and bottom friction although the small size of their published diagrams and the irregularities of the shape of the current ellipses would have concealed mean currents smaller than 5% of the tidal velocity amplitudes.

Retaining only the leading coriolis term, equation (3.4) becomes

$$\overline{C_y^*} = \Omega_z \frac{\partial \bar{u}_y}{\partial z} + kg \frac{\partial \bar{S}}{\partial x} \, . \qquad (3.5b)$$

3.4 <u>Vertical Terms</u>

The z component of equation (3.1) may be expanded in the same way as the y component. The convective terms may be written as

$$C_z = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u_x u_y + \frac{\partial}{\partial y} u_y u_y + \frac{\partial}{\partial z} u_y u_z\right) -$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} u_x u_x + \frac{\partial}{\partial y} u_x u_y + \frac{\partial}{\partial z} u_x u_z\right).$$

The ratio of the orders of magnitude of the first three terms to the second is $U_2 b_1/U_1\gamma$ and making the conservative assumption that $\gamma = 10b_1$ and $U_1 = 10U_2$ the first group of terms is seen to be negligible compared with the second. Averaging the resultant expression with respect to time and utilising equations (3.2) and (3.3) the following is obtained

$$\overline{C_z^*} = -\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \bar{u}_x\bar{u}_x + \frac{\partial}{\partial y} \bar{u}_x\bar{u}_y + \frac{\partial}{\partial z} \bar{u}_x\bar{u}_z\right) -$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \overline{U_xU_x} + \frac{\partial}{\partial y} \overline{U_xU_y} + \frac{\partial}{\partial z} \overline{U_xU_z}\right) -$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \overline{u'_xu'_x} + \frac{\partial}{\partial y} \overline{u'_xu'_y} + \frac{\partial}{\partial z} \overline{u'_xu'_z}\right). \qquad (3.6a)$$

As with equation (3.5a), in any particular case some of these terms may be negligible.

Again consideration of orders of magnitude shows that the local accelerations are negligible and that only the Ω_z coriolis term is significant, and hence the z component of equation (3.1) becomes

$$\overline{C_x^*} = \Omega_z \frac{\partial \bar{u}_z}{\partial z}. \qquad (3.6b)$$

3.5 Transverse Terms

Considering the x component of equation (3.1), the convective terms can be written as

$$C_x = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} u_xu_z + \frac{\partial}{\partial y} u_yu_z + \frac{\partial}{\partial z} u_zu_z\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} u_xu_y + \frac{\partial}{\partial y} u_yu_y + \frac{\partial}{\partial z} u_yu_z\right).$$

The ratio of the order of magnitude of the second group of terms to that of the first group is U_2b_1/U_3h, but it is not certain that one group is negligible compared with the other. One could, for example, construct a model estuary in which $U_2 << U_3$, and $b_1 < h$; however in a real estuary it is probable that $b_1 >> h$ and $U_2 \geqslant U_3$, and hence $U_2b_1/U_3h >> 1$. Thus the first group of terms may be neglected and hence

$$\overline{C_x^*} = -\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \bar{u}_x\bar{u}_y + \frac{\partial}{\partial y} \bar{u}_y\bar{u}_y + \frac{\partial}{\partial z} \bar{u}_y\bar{u}_z\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \overline{U_xU_y} + \frac{\partial}{\partial y} \overline{U_yU_y} + \frac{\partial}{\partial z} \overline{U_yU_z}\right)$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \overline{u'_xu'_y} + \frac{\partial}{\partial y} \overline{u'_yu'_y} + \frac{\partial}{\partial z} \overline{u'_yu'_z}\right). \qquad (3.7a)$$

Thus as before the x component of equation (3.1) can be approximated by

$$\overline{C_x^*} = \Omega_z \frac{\partial \bar{u}_x}{\partial z} - kg\,\frac{\partial \bar{S}}{\partial y}. \qquad (3.7b)$$

3.6 Salinity

We now consider the equation for salt conservation (2.3). The steady-state time-mean conservation equation can be written as

$$\nabla \cdot \overline{\underset{\sim}{u}S} = 0 \qquad (3.8)$$

which can be expanded to

$$\frac{\partial}{\partial x}(\bar{u}_x\bar{S}) + \frac{\partial}{\partial y}(\bar{u}_y\bar{S}) + \frac{\partial}{\partial z}(\bar{u}_z\bar{S}) + \frac{\partial}{\partial x}\overline{U_xS_T} + \frac{\partial}{\partial y}\overline{U_yS_T} + \frac{\partial}{\partial z}\overline{U_zS_T}$$
$$+ \frac{\partial}{\partial x}\overline{u'_xS'} + \frac{\partial}{\partial y}\overline{u'_yS'} + \frac{\partial}{\partial z}\overline{u'_zS'} = 0. \qquad (3.9)$$

In order to obtain this expression we have neglected terms of the type $\overline{\bar{u}_xS'}$, $\overline{U_xS'}$, $\overline{\bar{u}_xS_T}$ and $\overline{u'_x\bar{S}}$. It was shown in I that the tidal terms could be dropped in an elongated estuary in which the tide height was a small fraction of the depth, but this has not been shown to be generally true.

A possible relationship for tidal terms is a generalisation of that proposed in I:

$$\alpha = \left[\beta_\alpha \sin\eta(x - ct + \varepsilon_\alpha) + \gamma_\alpha \sin\eta(x + ct + \varepsilon_\alpha)\right] F_\alpha(z) G_\alpha(y) \qquad (3.10)$$

where $\alpha = S$, U_x, U_y or U_z and η, β, γ and ε are constants. Under certain conditions the tidal terms drop out of the equations of the model and under others they may be evaluated.

Equations (3.3), (3.5), (3.6), (3.7) and (3.9) together with relationships between the mean products of tidal and turbulence terms and appropriate boundary conditions form a very general model of estuarine flow. Such a general model cannot be solved analytically, but numerical

solutions are a possibility in the near future.

For the turbulence terms the usual introduction of eddy diffusivities K and eddy viscosities A effects a formal simplification and closes the system of equations. But if the law of variation of the eddy coefficients is not simple the substitution merely changes one problem for another. This fact has frequently led to excessively simple assumptions being made, but this of itself does not invalidate their use. The eddy coefficients may be defined as follows:

$$\overline{u'_m u'_n} = -A_{nm}\frac{\partial v}{\partial x_n} \qquad (3.11)$$

where v is the time mean velocity in the x_n direction, and

$$\overline{u'_n S'} = -K_n \frac{\partial \bar{S}}{\partial x_n}. \qquad (3.12)$$

Although this formulation does not specify the mode of variation of the eddy coefficients, it is usually assumed that they do not depend explicitly on the velocity distribution. If this is the case, equation (3.11) cannot be applied with complete generality.

4. Breadth Integrated Models

4.1 Integration across the breadth

The three-dimensional model derived above is clearly too complicated for analytic solutions or even numerical solutions with present computers. Furthermore for most cases many negligbly small terms would be retained quite unnecessarily, thus some simplification is essential. One of the most useful to consider is that lateral variations are less important than vertical or longitudinal variations. A sequence of two-dimensional models is developed below, with increasingly restrictive assumptions. The first model involves integration of the complete equations of motion over the breadth. If all terms are retained the integrated equations are exact, but for reasons given above terms are dropped and approximate integrated equations are obtained.

Before making these simplifications a kinematic boundary condition will be introduced. The condition for the flow to be tangential to the sides of the estuary is

$$[u_y]_\alpha = + \frac{\partial \alpha}{\partial x} [u_x]_\alpha + \frac{\partial \alpha}{\partial z} [u_z]_\alpha \quad \text{for } \alpha = a,b. \tag{4.1}$$

where the sides of the channel are given by: $y = a\ (x,z)$ and $y = b\ (x,z)$. Equation (4.1) may be multiplied by $[\beta]_\alpha$ and then the time mean taken to give

$$[\overline{\beta\, u_y}]_\alpha = + \frac{\partial \alpha}{\partial x} [\overline{\beta u_x}]_\alpha + \frac{\partial \alpha}{\partial z} [\overline{\beta u_z}]_\alpha \tag{4.2}$$

where β could be u_x, u_y, u_z, S or any other scalar or vector.

By the application of Leibnitz's rule, equation (3.8) may be integrated over the breadth from $y = a(x,z)$ to $y = b(x,z)$ to give

$$0 = \int_a^b \nabla \cdot \overline{\underset{\sim}{u}S}\, dy$$

$$= \frac{\partial}{\partial x} \int_a^b \overline{u_x S}\, dy + \frac{\partial a}{\partial x} [\overline{u_x S}]_a - \frac{\partial b}{\partial x} [\overline{u_x S}]_b + [\overline{u_y S}]_a^b$$

$$+ \frac{\partial}{\partial z} \int_a^b \overline{u_z S}\, dy + \frac{\partial a}{\partial z} [\overline{u_z S}]_a - \frac{\partial b}{\partial z} [\overline{u_z S}]_b$$

By applying equation (4.2) some terms may be eliminated to give

$$0 = \frac{\partial}{\partial x} (B \langle \overline{u_x S} \rangle) + \frac{\partial}{\partial z} (B \langle \overline{u_z S} \rangle)$$

$$= \frac{\partial}{\partial x} (B \langle \bar{u}_x \bar{S} \rangle) + \frac{\partial}{\partial x} (B \langle \overline{U_x S_T} \rangle)$$

$$+ \frac{\partial}{\partial x} (B \langle \overline{u_x' S'} \rangle) + \frac{\partial}{\partial z} (B \langle \bar{u}_z \bar{S} \rangle)$$

$$+ \frac{\partial}{\partial z} (B \langle \overline{U_z S_T} \rangle) + \frac{\partial}{\partial z} (B \langle \overline{u_z' S'} \rangle) \tag{4.3}$$

where the average over the breadth is indicated by < > and B (x,z) = b (x,z) - a (x,z) is the breadth. This result is equivalent to an equation first derived for estuaries by Pritchard (10) and in more detail by Okubo (11).

To obtain spatially averaged vorticity equations it is preferable not to integrate equations (3.5), (3.6) and (3.7) over the breadth. This is because the integrals of the terms omitted in obtaining these equations from equation (3.1) might not all be negligible. If the complete vorticity equations are integrated, the following expressions are obtained for the convective terms by use of equation (4.2), since $\nabla\rho \times (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}$ may still be neglected:

$$B \langle \bar{C}_y \rangle = \frac{\partial a}{\partial z} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_x})]_a - \frac{\partial b}{\partial z} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_x})]_b + \frac{\partial}{\partial z} [\frac{\partial}{\partial x} B \langle \overline{u_x u_x} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_x u_z} \rangle]$$

$$- \frac{\partial a}{\partial x} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_z})]_a + \frac{\partial b}{\partial x} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_z})]_b$$

$$- \frac{\partial}{\partial x} [\frac{\partial}{\partial x} B \langle \overline{u_x u_z} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_z u_z} \rangle] , \qquad (4.4a)$$

$$B \langle \bar{C}_x \rangle = - \frac{\partial a}{\partial z} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_y})]_a + \frac{\partial b}{\partial z} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_y})]_b + [\nabla\cdot(\overline{\underset{\sim}{u}\, u_z})]_a^b$$

$$- \frac{\partial}{\partial z} [\frac{\partial}{\partial x} B \langle \overline{u_y u_x} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_y u_z} \rangle], \qquad (4.5a)$$

$$B \langle \bar{C}_z \rangle = \frac{\partial a}{\partial x} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_y})]_a - \frac{\partial b}{\partial x} [\nabla\cdot(\overline{\underset{\sim}{u}\, u_y})]_b - [\nabla\cdot(\overline{\underset{\sim}{u}\, u_x})]_a^b$$

$$+ \frac{\partial}{\partial x} [\frac{\partial}{\partial x} B \langle \overline{u_x u_y} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_y u_z} \rangle] . \qquad (4.6a)$$

If it is supposed

(i) that the length scales for variation in $\underset{\sim}{\Omega}$ are very much larger than those for $\underset{\sim}{\bar{u}}$, so that $\underset{\sim}{\Omega}$ can be taken outside the integral,

(ii) that the length scale for $\underset{\sim}{\bar{u}}$ in the x direction is very much larger than that in the z direction, and

(iii) that Ω_x, Ω_y, Ω_z are of the same order of magnitude,

and noting that $\nabla\rho \times (\underset{\sim}{u} \times \underset{\sim}{\Omega})$ may still be neglected, it can be shown that the integral across the estuary of the coriollis term reduces to

$$\int_a^b \nabla \times (\underset{\sim}{\bar{u}} \times \underset{\sim}{\Omega})\, dy$$

$$= \hat{i}\{[\Omega_y\ \bar{u}_x]_a^b + \Omega_z\ ([\frac{\partial a}{\partial z}\ \bar{u}_x]_a - \frac{\partial b}{\partial z} [\bar{u}_x]_b + \frac{\partial}{\partial z} B \langle \bar{u}_x \rangle)\}$$

$$+ \hat{j}\{[\Omega_y\ \bar{u}_y]_a^b + \Omega_z\ ([\frac{\partial a}{\partial z}\ \bar{u}_y]_a - \frac{\partial b}{\partial z} [\bar{u}_y]_b + \frac{\partial}{\partial z} B \langle \bar{u}_y \rangle)\}$$

$$+ \hat{k}\{[\Omega_y\ \bar{u}_z]_a^b + \Omega_z\ ([\frac{\partial a}{\partial z}\ \bar{u}_z]_a - \frac{\partial b}{\partial z} [\bar{u}_z]_b + \frac{\partial}{\partial z} B \langle \bar{u}_y \rangle)\} .$$

Integration of the remaining terms in the complete equation (3.1) does not yield any significant terms not obtained by integrating equations (3.5b), (3.6b) and (3.7b). Integration of these last three equations with the substitution of the above expression for coriolis forces gives:

$$B \langle \bar{C}_y \rangle = \Omega_z \frac{\partial}{\partial z} (B \langle \bar{u}_y \rangle) + kg \frac{\partial B \langle \bar{S} \rangle}{\partial x} - \Omega_z \frac{\partial b}{\partial z} [\bar{u}_y]_b$$

$$+ \Omega_z \frac{\partial a}{\partial z} [\bar{u}_y]_a + [\Omega_y \bar{u}_y]_a^b$$

$$- kg \frac{\partial b}{\partial x} [\bar{S}]_b + kg \frac{\partial a}{\partial x} [\bar{S}]_a , \qquad (4.4b)$$

$$B \langle \bar{C}_z \rangle = \Omega_z \frac{\partial B \langle \bar{u}_z \rangle}{\partial z} - \Omega_z \frac{\partial b}{\partial z} [\bar{u}_z]_b + \Omega_z \frac{\partial a}{\partial z} [\bar{u}_z]_a$$

$$+ [\Omega_y \bar{u}_y]_a^b ; \qquad (4.6b)$$

$$B \langle \bar{C}_x \rangle = \Omega_z \frac{\partial B \langle \bar{u}_x \rangle}{\partial z} - [kg\bar{S}]_a - \Omega_z \frac{\partial b}{\partial z} [\bar{u}_x]_b$$

$$+ \Omega_z \frac{\partial a}{\partial z} [\bar{u}_x]_a \quad . \qquad (4.5b)$$

The continuity equation (3.3a) may also be integrated across the breadth to give:

$$0 = \int_a^b \nabla \cdot \bar{\underline{u}} \, dy$$

$$= \frac{\partial}{\partial x} B \langle \bar{u}_x \rangle + \frac{\partial a}{\partial x} [\bar{u}_x]_a - \frac{\partial b}{\partial x} [\bar{u}_x]_b + [\bar{u}_y]_a^b$$

$$- \frac{\partial}{\partial z} B \langle \bar{u}_z \rangle + \frac{\partial a}{\partial z} [\bar{u}_z]_a - \frac{\partial b}{\partial z} [\bar{u}_z]_b,$$

and by use of (4.1) this becomes

$$0 = \frac{\partial}{\partial x} B \langle \bar{u}_x \rangle + \frac{\partial}{\partial z} B \langle \bar{u}_z \rangle \qquad (4.7a)$$

and similarly (3.3b) gives

$$0 = \frac{\partial}{\partial x} B \langle U_x \rangle + \frac{\partial}{\partial z} B \langle U_z \rangle \, . \qquad (4.7b)$$

4.2 The Simplified Breadth-integrated Equations

The equations (4.3) to (4.7) obtained by integration over the breadth of the estuary are very general and also rather complex. They are therefore of limited value, and it is necessary to consider ways of simplifying them. Two simplifications are possible, the first being to apply the same order of magnitude argument as used in the previous section. This justifies the omission of half the terms in equation (4.4a), but will not be applied to equations (4.5a) and (4.6a) because there is not such a

large margin between the major and minor terms in these equations as in equation (4.4a).

The second simplification results from applying a second boundary condition at the sides of the estuary. The simplest is that there is no slip between the sides and the water at the sides, and this together with the condition of no normal flow at the sides is equivalent to

$$[\underline{u}]_a = [\underline{u}]_b = 0.$$

This condition is physically realistic and gives rise to the boundary layer on the sides of the estuary which is important in analysis of estuarine flushing, but its use leads to two difficulties. The first is that viscous shear stresses have been assumed to be contained in the Reynolds stress terms, and while the velocity fluctuations vanish at the boundaries the shear stress does not. This difficulty may be completely overcome by not setting the Reynolds stresses equal to zero; hence the boundary condition to be imposed becomes

$$[\bar{u}_x, \bar{u}_y, \bar{u}_z, U_x, U_y, U_z]_a = [\bar{u}_x, \bar{u}_y, \bar{u}_z, U_x, U_y, U_z]_b = 0. \qquad (4.8)$$

The second disadvantage of this boundary condition is that it may conflict with the subsequently assumed expressions for the eddy viscosities, which in order to be of use must be simple functions valid over the bulk of the flow. Although functions which are too simple cannot give correct shear stresses near the boundaries as well as in the remainder of the fluids, Johns (12), (13) obtained satisfactory results with reasonably simple expressions for A_{xz}. The assumed functions would have to be sufficiently accurate at the boundaries as well as over the bulk of the flow.

By applying equation (4.8) all quantities in equation (4.4a) evaluated at the boundaries are set equal to zero except for the Reynolds stresses. The following expressions is then obtained by supposing that $\gamma >> h$;

$$B \langle \bar{C}_y \rangle = \frac{\partial}{\partial z} \left[\frac{\partial}{\partial x} B \langle \overline{u_x u_x} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_x u_z} \rangle\right]$$

$$+ \frac{\partial a}{\partial z} \left[\frac{\partial}{\partial x} (\overline{u'_x u'_x}) + \frac{\partial}{\partial y} (\overline{u'_x u'_y}) + \frac{\partial}{\partial z} (\overline{u'_x u'_z})\right]_a$$

$$- \frac{\partial b}{\partial z} \left[\frac{\partial}{\partial x} (\overline{u'_x u'_x}) + \frac{\partial}{\partial y} (\overline{u'_x u'_y}) + \frac{\partial}{\partial z} (\overline{u'_x u'_z})\right]_b .$$

Although length scales for the turbulent velocity components could be expected to be more or less equal, the length scales for time-mean products of velocities will not and will probably be the same as those for mean velocities (γ, b_1, h). Presuming that $\gamma >> b_1$,h the first of the Reynolds stress terms in each bracket including the terms which have not been expanded may be neglected in comparison with the others. The expression thus becomes:

$$B \langle \bar{C}_y \rangle = \frac{\partial}{\partial z} \left[\frac{\partial}{\partial x} B \langle \overline{u_x u_x} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_x u_z} \rangle\right]$$

$$+ \frac{\partial a}{\partial z} \left[\frac{\partial}{\partial y} (\overline{u'_x u'_y}) + \frac{\partial}{\partial z} (\overline{u'_x u'_z})\right]_a$$

$$- \frac{\partial b}{\partial z} \left[\frac{\partial}{\partial y} (\overline{u'_x u'_y}) + \frac{\partial}{\partial z} (\overline{u'_x u'_z})\right]_b . \qquad (4.9a)$$

Similarly equations (4.5a) and (4.6a) become

$$B \langle \bar{C}_x \rangle = - \frac{\partial}{\partial z} \left[\frac{\partial}{\partial x} B \langle \overline{u_y u_x} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_y u_z} \rangle\right]$$

$$- \frac{\partial a}{\partial z} \left[\frac{\partial}{\partial y} \overline{u'_y u'_y} + \frac{\partial}{\partial z} \overline{u'_z u'_y}\right]_a$$

$$+ \left[\frac{\partial}{\partial y} \overline{u'_y u'_z} + \frac{\partial}{\partial z} \overline{u'_z u'_z}\right]_a^b$$

$$+ \frac{\partial b}{\partial z} \left[\frac{\partial}{\partial y} \overline{u'_y u'_y} + \frac{\partial}{\partial z} \overline{u'_z u'_y}\right]_b , \qquad (4.9b)$$

$$B \langle \bar{C}_z \rangle = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} B \langle \overline{u_x u_y} \rangle + \frac{\partial}{\partial z} B \langle \overline{u_y u_z} \rangle\right]$$

$$- \left[\frac{\partial}{\partial y} \overline{u'_x u'_y} + \frac{\partial}{\partial z} \overline{u'_x v'_z}\right]_a^b \qquad (4.9c)$$

where, in obtaining equation (4.9c), the condition $\gamma >> h$ was used to eliminate boundary stress terms containing the factors $\frac{\partial a}{\partial x}$ or $\frac{\partial b}{\partial x}$ by comparison with those retained.

Equations (4.4b), (4.5b) and (4.6b) may be combined with equations (4.9) and the boundary condition (4.8). After the expansion of the time mean terms, the following expressions are obtained:

$$B \langle \bar{C}_y \rangle = \frac{\partial}{\partial z} \Big[\frac{\partial}{\partial z} (B \langle \bar{u}_x \bar{u}_x \rangle) + \frac{\partial}{\partial x} (B \langle \overline{U_x U_x} \rangle) + \frac{\partial}{\partial z} (B \langle \bar{u}_x \bar{u}_z \rangle) + \frac{\partial}{\partial z} (B \langle \overline{U_x U_z} \rangle)$$

$$+ \frac{\partial}{\partial z} (B \langle \bar{u}'_x \bar{u}'_z \rangle)\Big] + \frac{\partial a}{\partial z} \Big[\frac{\partial}{\partial y} \overline{u'_x u'_y} + \frac{\partial}{\partial z} \overline{u'_x u'_z}\Big]_a$$

$$- \frac{\partial b}{\partial z} \Big[\frac{\partial}{\partial y} \overline{u'_x u'_y} + \frac{\partial}{\partial z} \overline{u'_x u'_z}\Big]_b$$

$$= \Omega_z \frac{\partial}{\partial z} B \langle \bar{u}_y \rangle + kg \frac{\partial}{\partial x} B \langle \bar{S} \rangle - kg \frac{\partial b}{\partial x} [\bar{S}]_b + kg \frac{\partial a}{\partial x} [\bar{S}]_a , \qquad (4.10a)$$

$$B \langle \bar{C}_x \rangle = - \frac{\partial}{\partial z} \Big[\frac{\partial}{\partial x} B \langle \bar{u}_y \bar{u}_x \rangle + \frac{\partial}{\partial x} B \langle \overline{U_y U_x} \rangle + \frac{\partial}{\partial z} B \langle \bar{u}_y \bar{u}_z \rangle + \frac{\partial}{\partial z} B \langle \overline{U_y U_z} \rangle$$

$$+ \frac{\partial}{\partial z} B \langle \overline{u'_y u'_z} \rangle\Big] - \frac{\partial a}{\partial z} \Big[\frac{\partial}{\partial y} \overline{u'_y u'_y} + \frac{\partial}{\partial z} \overline{u'_z u'_y}\Big]_a$$

$$+ \frac{\partial b}{\partial z} \Big[\frac{\partial}{\partial y} \overline{u'_y u'_y} + \frac{\partial}{\partial z} \overline{u'_z u'_y}\Big]_b + \Big[\frac{\partial}{\partial y} \overline{u'_y u'_z} + \frac{\partial}{\partial z} \overline{u'_z u'_z}\Big]_a^b$$

$$= \Omega \frac{\partial}{\partial z} B \langle \bar{u}_x \rangle - kg [\bar{S}]_a^b , \qquad (4.10b)$$

$$B \langle \bar{C}_z \rangle = \frac{\partial}{\partial z} \Big[\frac{\partial}{\partial x} B \langle \bar{u}_x \bar{u}_y \rangle + \frac{\partial}{\partial x} \langle \overline{U_x U_y} \rangle + \frac{\partial}{\partial z} B \langle \bar{u}_y \bar{u}_z \rangle + \frac{\partial}{\partial z} \langle \overline{U_y U_z} \rangle$$

$$+ \frac{\partial}{\partial z} B \langle \overline{u'_y u'_z} \rangle\Big] - \Big[\frac{\partial}{\partial y} \overline{u'_x u'_y} + \frac{\partial}{\partial z} \overline{u'_x u'_z}\Big]_a^b$$

$$= \Omega_z \frac{\partial}{\partial z} B \langle \bar{u}_z \rangle . \qquad (4.10c)$$

The integrated model obtained comprises equations (4.3), (4.7) and (4.10) together with boundary conditions and closure relationships such as equations (3.10), (3.11) and (3.12). Despite the simplifications this model is still intractible and is unlikely to be used directly in analytical studies. Its most likely uses will be as a basis for simpler

models for analysis, in numerical models and in the reduction and evaluation of experimental data.

The latter use is well illustrated by the experimental studies referred to in Part I. In many of these studies one or two terms which were not measured were evaluated by the use of approximate equations, but in fact the values given for these terms contained the sum of all the unmeasured terms. The usual techniques of measurement of velocity and salinity would enable all the terms except the turbulence fluctuations to be directly measured. In the absence of measurements of turbulence the equations of the model would furnish four relationships for two turbulent momentum fluxes, two turbulent salt fluxes and ten boundary shear stress quantities. Use of equations such as (3.11) and (3.12) or use of results from other turbulent flow situations would be necessary if any terms must be explicitly evaluated from measurements of non-turbulent quantities.

4.3 The Breadth Integrated Model

In order to develop an essentially two-dimensional model which might be suitable for analytical studies it has been common practice to assume that none of the variables and parameters of the flow depend upon the transverse coordinate and that $\bar{u}_y = 0$. This assumption is unnecessarily restrictive as may be seen from the following descriptive argument. The flow at an expanding section of an estuary may be regarded as having originated at a distant point source, as in Fig. 1.

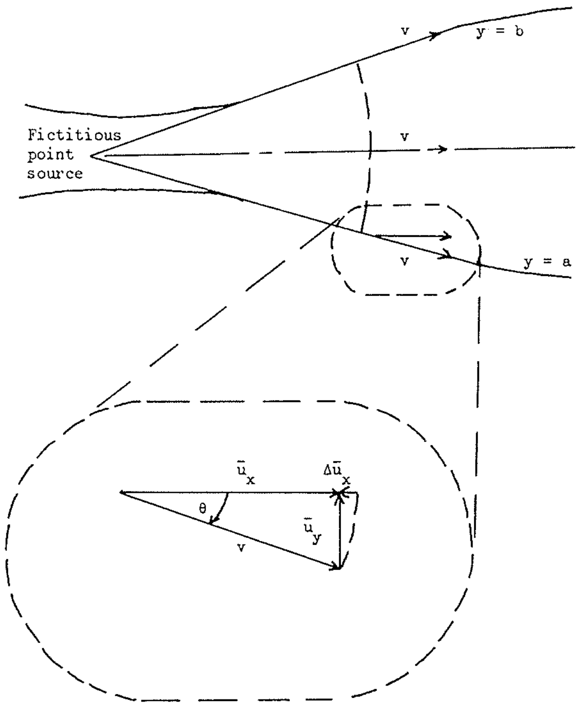

Fig. 1. Plan View of Divergent Section of Estuary

At the centreline $\bar{u}_x = v$, but near the edges $\bar{u}_x = v + \Delta\bar{u}_x$

where $\Delta\bar{u}_x/\bar{u}_y = \bar{u}_y/\bar{u}_x = \tan\theta = \frac{\partial b}{\partial x} = \frac{1}{2}\frac{\partial B}{\partial x}$,

hence $$\frac{\Delta\bar{u}_x}{\bar{u}_x} = \left(\frac{\partial b}{\partial x}\right)^2 ,$$

and $$O\left(\frac{\Delta\bar{u}_x}{\bar{u}_x}\right) = \left(\frac{\partial B}{\partial x}\right)^2 ,$$

and $$O\left(\frac{\bar{u}_y}{\bar{u}_x}\right) = \left(\frac{\partial B}{\partial x}\right) .$$

The assumption that $\bar{u}_x$ is constant across the breadth introduces an error of order $(\frac{\partial B}{\partial x})^2$ and hence requires that $(\frac{\partial B}{\partial x})^2 \ll 1$. The usual assumption that $\bar{u}_y$ is negligible everywhere requires that in the continuity equation

$$\frac{\partial\bar{u}_y}{\partial y} \ll \frac{\partial\bar{u}_x}{\partial x}$$

i.e. $$O\left(\frac{\bar{u}_y}{\bar{u}_x}\right) \ll \frac{B}{\gamma}$$

and hence that $(\frac{\partial B}{\partial x}) \ll \frac{B}{\gamma}$, i.e. that the breadth changes by a negligible fraction. A similar argument may be presented for tidal velocities.

Although not exact, this argument shows that the assumption of truly two-dimensional flow is restrictive, and so this assumption will not be imposed. Instead it will be assumed that none of the variables which appear as spatially-averaged quantities in equations (4.3), 4.7) and (4.10) do vary with y. This assumption is equivalent to the following:

$$\langle \bar{u}_\alpha \bar{S} \rangle = \bar{u}_\alpha \bar{S}, \quad \langle \overline{U_\alpha S_T} \rangle = \overline{U_\alpha S_T},$$

$$\langle \overline{u'_\alpha S'} \rangle = \overline{u'_\alpha S'}, \quad \langle \bar{u}_\alpha \bar{u}_\beta \rangle = \bar{u}_\alpha \bar{u}_\beta,$$

$$\langle \overline{U_\alpha U_\beta} \rangle = \overline{U_\alpha U_\beta}, \quad \langle \overline{u'_\alpha u'_\beta} \rangle = \overline{u'_\alpha u'_\beta}$$

where α and β stand for either x or z.

The assumption just made does not apply at the walls to the derivatives of the Reynolds stresses. The evaluation of these quantities, which requires the use of information additional to the above three equations, does not form the principal part of the analysis. Indeed, in many cases order of magnitude arguments may be applied to eliminate many of these terms.

Before equation (4.10a) can be rewritten, it is necessary to refer to section 3 where it was shown that the coriolis terms could be neglected in most cases. In terms of the magnitudes of section 3, the condition to be satisfied if this term is to be neglected may be expressed as

$$\Omega \, U_2 \ll \max \{khg \, S_o/\gamma, \; U_1^2/\gamma\}$$

where max indicates that the larger of the terms is to be chosen. The first term in the bracket is the order of magnitude of the gravitational term while the second is that of the inertial terms.

If this condition is met, equation (4.10a) becomes

$$\frac{\partial}{\partial z}\Big[\frac{\partial}{\partial x}(B\,\bar{u}_x\bar{u}_x) + \frac{\partial}{\partial x}(B\,\overline{U_x U_x}) + \frac{\partial}{\partial z}(B\,\bar{u}_x\bar{u}_z)$$

$$+ \frac{\partial}{\partial z}(B\,\overline{U_x U_z}) + \frac{\partial}{\partial z}(B\,\overline{u'_x u'_z})\Big] + G$$

$$= kg\,\frac{\partial a}{\partial x}[\bar{S}]_a - kg\,\frac{\partial b}{\partial x}[\bar{S}]_b + kg\,\frac{\partial}{\partial x}(B\bar{S}) \qquad (4.11a)$$

where

$$G = \frac{\partial a}{\partial z}\Big[\frac{\partial}{\partial y}(\overline{u'_x u'_y}) + \frac{\partial}{\partial z}(\overline{u'_x u'_z})\Big]_a$$

$$- \frac{\partial b}{\partial z}\Big[\frac{\partial}{\partial y}(\overline{u'_x u'_y}) + \frac{\partial}{\partial z}(\overline{u'_x u'_z})\Big]_b \qquad (4.11b)$$

and the brackets < > have been omitted for convenience. In an uncurved reach of an estuary

$$G = -\,2\,\frac{\partial B}{\partial z}\Big[\frac{\partial}{\partial y}(\overline{u'_x u'_y}) + \frac{\partial}{\partial z}(\overline{u'_x u'_z})\Big]_b$$

and the right hand side of (4.11a) reduces to $kg\,B\,\frac{\partial \bar{S}}{\partial x}$.

The salinity equation was simplified in Part I by eliminating the tidal terms. This follows from the fact that the tidal velocities are out of phase with the salinity (Pritchard (1)) and has been supported by the data of Hinwood (14) and Dyer and Ramamoorthy (15). This simplification should be quite generally valid provided that the variation of the tidal currents and salinities about the mean do not show marked asymmetries over a tidal cycle. Equation (4.3) thus becomes:

$$\frac{\partial}{\partial x}(B\,\bar{u}_x\bar{S}) + \frac{\partial}{\partial z}(B\,\bar{u}_z\bar{S}) + \frac{\partial}{\partial x}(B\,\overline{u'_x S'}) + \frac{\partial}{\partial z}(B\,\overline{u'_z S'}) = 0\,. \qquad (4.12)$$

The basic two-dimensional model consists of equations (4.7), (4.11a) and (4.12), in which all mean quantities are treated as invariant with breadth, and the operator < > has been dropped for convenience, together with equation (4.11b), boundary conditions and closure relationships such as those already mentioned.

Although we do not intend to treat boundary conditions, for completeness some mention of them must be made. We have already introduced two conditions on the sides of the estuary: the condition of zero normal velocity (4.1) and the condition of zero velocity (4.8). The only other condition to be applied on the sides is to specify the Reynolds stress terms - in practice to express these turbulent quantities in terms of the mean motion. At the bed of the estuary the same conditions as on the sides may be applied, with appropriate changes of symbols.

The free surface may be subject to shear stress, produced by the wind, or to turbulent energy input from wind waves. As with all Reynolds stresses these terms must be parameterised. The model permits free surface elevation changes, but the kinematic and energy boundary conditions have not been stated, they are the usual conditions for gravity waves. Depending on the technique of solution adopted these may not be necessary.

At the upstream and downstream ends of the estuary it is not clear how many boundary conditions are required, but by analogy with the physical system we could prescribe, as functions of time, the surface level and the vertical distribution of salinity at landward and seaward ends, and the river flow. The disappearance of the velocity seawards could also be a valid requirement.

4.4 An Approximate Model for Estuaries of Rectangular Section.

We now show how the model consisting of equations (4.7), (4.11) and (4.12) can be further simplified when the flow is restricted to being in a channel of rectangular cross-section and some additional approximations are made in order to remove the convective term. In the end we obtain the model which was analysed by Hansen and Rattray and discussed in Part I.

It was shown in Appendix 1 of Part I that the tidal term can be ignored provided one of the following conditions is satisfied

(i) the tide is a pure progressive wave, or

(ii) there is no significant variation of tidal velocity with depth, or

(iii) the length of the tidal wave is much greater than that of the estuary.

In the remaining part of this section we assume that one of these conditions is satisfied. For an estuary of rectangular cross section $\frac{\partial a}{\partial z} = \frac{\partial b}{\partial z} = \frac{\partial B}{\partial z} = 0$ and hence $G = 0$, and since the same order of magnitude arguments apply even in this somewhat artificial case there is no need to restore terms omitted in going from equation (4.4a) to (4.11). The convective terms may be dropped if the Froude number $F = \bar{U}_1/\sqrt{g\, kS_o h} \ll 1$ since the ratio of the convective terms to the gravitational term is:

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x} B\, \bar{u}_x \bar{u}_x + \frac{\partial}{\partial z} B\, \bar{u}_x \bar{u}_z\right] / kg\, B \frac{\partial \bar{S}}{\partial x}$$

which is of order F throughout the portion of the estuary for which the orders of magnitude chosen are correct. Pritchard's data for the James River (Pritchard (2), Table 2) showed the convective terms to be negligible, but his methods of evaluating the quantities which he did not measure are questionable. If $F \ll 1$, equation (4.11) is reduced to

$$\frac{\partial^2}{\partial z^2} B\, \overline{u'_x u'_z} = kg\, B \frac{\partial \bar{S}}{\partial x} . \qquad (4.13)$$

The continuity equation (4.7a) is satisfied by a stream function Ψ defined by:

$$\frac{\partial \Psi}{\partial z} = - B\, \bar{u}_x, \quad \frac{\partial \Psi}{\partial x} = B\, \bar{u}_z$$

Then by using equations (3.11) and (3.12) equation (4.13) becomes

$$\frac{\partial^2}{\partial x^2}\left(A \frac{\partial^2 \Psi}{\partial z^2}\right) = k\, B\, g \frac{\partial \bar{S}}{\partial x} \qquad (4.14)$$

and equation (4.12) becomes

$$\frac{\partial \Psi}{\partial x}\frac{\partial \bar{S}}{\partial z} - \frac{\partial \Psi}{\partial z}\frac{\partial \bar{S}}{\partial x} = \frac{\partial}{\partial x}\left(B\, K_x \frac{\partial \bar{S}}{\partial x}\right) + \frac{\partial}{\partial z}\left(B\, K_z \frac{\partial \bar{S}}{\partial z}\right). \qquad (4.15)$$

The simplified model consisting of equations (4.14) and (4.15) together with boundary conditions formed the starting point for analyses such as those reported in I and II. It is a valid approximation only if the assumptions set out in the course of this derivation are satisfied. An immediate restriction is imposed by the convective terms, and so application of the model should be restricted to slightly stratified estuaries similar in regime to the James River, with low values of F.

5. Concluding Remarks

In section 3, by making minor simplifications, we obtained a very general mathematical model of estuarine flow. This model consists of the time-averaged and tidal continuity equations, the time-averaged salt conservation equation and the three components of the time-averaged vorticity equation. For closure this system requires relationship between turbulent and mean terms, and boundary conditions must be specified. At the moment this model is too complex for solution and is regarded as the starting point for the development of simpler models.

By integrating across the breadth, and using two boundary conditions an integrated model has been obtained and then by order of magnitude analysis this has been simplified. This model has three functions: as the starting point in the derivation of simpler models, as the basis of a numerical model, and in the evaluation of experimental data.

For a rectangular estuary additional assumptions and approximations lead to the model analysed by Hansen and Rattray. The derivation shows the restrictions on the use of this model. Less restricted models have not been solved analytically, but could be studied with a view to obtaining asymptotic solutions or providing bounds on the solutions.

Future work will be concerned with completing the specification of the general model and with deducing meaningful simplified models.

To complete the specification boundary conditions must be obtained, but because of the nonlinearity of the problem it is not clear how many are required in a given case and a preferred sequence of boundary conditions may have to be given so that sufficient may be chosen to specify the problem without constraint. Closure requires relationships between the mean and turbulent terms, and this seems to require measurements in actual estuaries. Such measurements are in progress.

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