A new method to study steady unconfined flow through porous media

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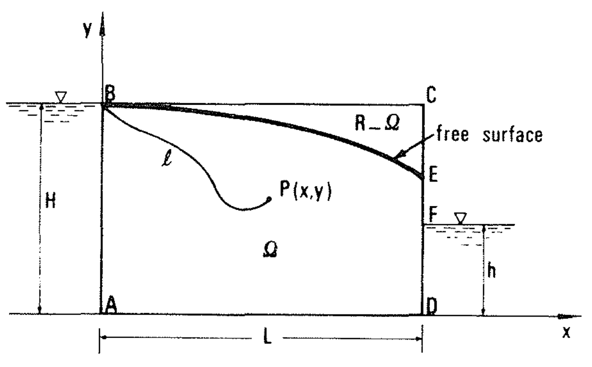
1. As it is clearly said in the title, I will talk very rapidly about the principles of a new mathematical method to analyse steady unconfined flow through porous media, which has been found by Baiocchi [1] and represents the first results of a scientific collaboration between the Laboratory of Numerical Analysis of the National Council on Research and the Hydraulic Institute of the University of Pavia.

To make the matter more understandable, I will illustrate the application of this method to the most simple of such problems, that is the classical one of the flow above a horizontal sole and between parallel vertical boundaries (fig.1).

Indicating with R the rectangular dam $ABCD$ and with Ω the part of it through which the water flows, the method consists, first of all, in the substitution of the traditional variable *piezometric head:*



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$$u\,(x,y) = y + \frac{p}{\gamma} \qquad (1)$$

with the variable $w(x,y)$ defined in the rectangle R this way:

$$w(x,y) = \int_l \left[-\tilde{v}\,dx + (y - \tilde{u})\,dy \right] \qquad (2)$$

where the integration is made along a generic line l that links the generic point $P(x,y)$ with the starting point B of the free line, in the BP direction; furthermore, indicating with v the stream function, the symbols $\tilde{v}$ and $\tilde{u}$ represents the prolongations of v and u beyond the wet region Ω and they are defined this way:

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ y & \text{in } R - \Omega \end{cases} \qquad \tilde{v} = \begin{cases} v & \text{in } \Omega \\ o & \text{in } R - \Omega \end{cases} \qquad (3)$$

It is easy to see that:

$$-\tilde{v}\,dx + (y - \tilde{u})\,dy$$

is an exact differential; it follows thus that the value of $w(x,y)$ doesn't depend on the trajectory that joins the points P and B (1).

From relations (2) and (3) follows that $w > 0$ in the domain Ω and $w = 0$ in the domain $R - \Omega$; thus we immediately see that, once we have obtained in all the points of the rectangle the values of w, the flow is completely known and the

(1) It is interesting to note that, if we choose as the trajectory the $BP'P$ one indicated in fig. 2, and if we put conventionally $v = 0$ on the freeline, given that on this line is:

$$y - u = 0$$

we have

$$w(x,y) = \int_h^y (t-u)\,dt = \int_y^h (u-t)\,dt = \frac{1}{\gamma}\int_y^h p\,dt$$

that is $w(x,y)$ is proportional to the total force upon the plane vertical surface above the point P.

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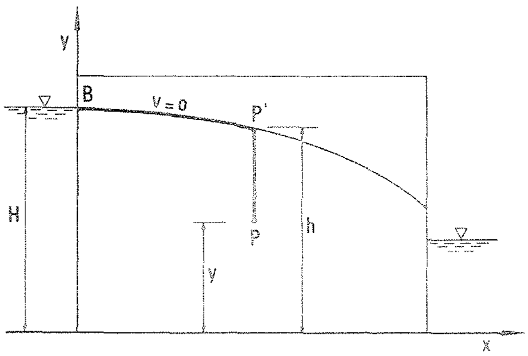

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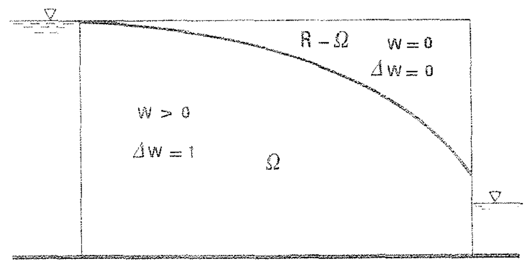

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freeline is individuated as the element of separation between the zone where $w > 0$ and the one where $w = 0$.

The piezometric head $u(x,y)$ is obtained from the relation:

$$u(x,y) = y - \frac{\partial w}{\partial y} \qquad (4)$$

that we immediately obtain from (2).

From the definition of $w(x,y)$ it follows that the stream function is equal and of opposite sign to $\frac{\partial w(x,y)}{\partial x}$; because of that, the streamlines can be obtained by drawing the curves $\frac{\partial w}{\partial x}$ = constant.

To calculate $w(x,y)$, we begin by translating the known relations on $u(x,y)$ into the relations on $w(x,y)$.

From the relations (2) and (3), given that:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

we easily obtein:

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } R - \Omega \end{cases} = \chi_\Omega$$

where χ_Ω is the characteristic function of Ω.

Thus this problem can be formulated this way (fig. 3):

$$\begin{matrix} w > 0, & \Delta w = 1 & \text{in } \Omega \\ w = 0, & \Delta w = 0 & \text{in } R - \Omega \end{matrix} \qquad (5)$$

These relations, considering the fact that on the free line Δw hasn't concentrated mass, it is possible to write synthetically:

$$w \geq 0 \qquad 1 - \Delta w \geq 0 \qquad w(1 - \Delta w) \equiv 0 \qquad (6)$$

in the whole rectangle.

To be able to completely individuate $w(x,y)$, it is from now on enough to know the w values on the boundaries ∂R of the rectangle.

These can be determined by noting that (fig. 1):

$u = H$	along AB
$u = y$	along BEF
$u = h$	along FD
$\frac{\partial u}{\partial y} = 0$	along AD
$\frac{\partial u}{\partial n} = 0$	along BE

Indicating by g the values that w assumes on ∂R, we get (fig. 4):

- along AB:

$$g(0,y) = \int_H^y (t - H)\, dt = \frac{1}{2} (H - y)^2$$

- along BC:

$$g(x,H) = - \int_0^x \tilde{v}\, dt = 0$$

- along CEF:

$$g(l,y) = - \int_0 \tilde{v}\, dt + \int_H^y (t - \tilde{u})\, dt = 0$$

given that:

$$\begin{cases} \tilde{v} = 0 & \text{along } BC \\ \tilde{u} = y & \text{along } CEF \end{cases}$$

- along FD

$$g(l,y) = - \int_0^l \tilde{v}\, dt + \int_H^h (t - \tilde{u})\, dt + \int_h^y (t - \tilde{u})\, dt = \frac{1}{2}(h-y)^2$$

given that

$\tilde{v} = 0$	along BC
$\tilde{u} = y$	along CEF
$\tilde{u} = h$	along FD

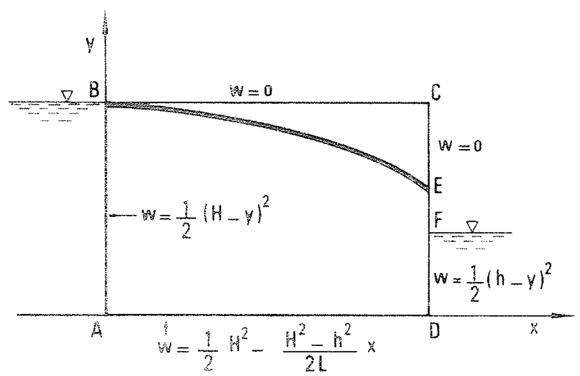
- along DA

since, as a convention, $w = 0$ is on the free line, it turns out that on DA is:

$$v = \frac{q}{k} ,$$

where we have indicated with q the discharge and with k the coefficient of permeability; therefore we have:

$$\left(\frac{\partial w}{\partial x}\right)_{AD} = - \frac{q}{k}$$



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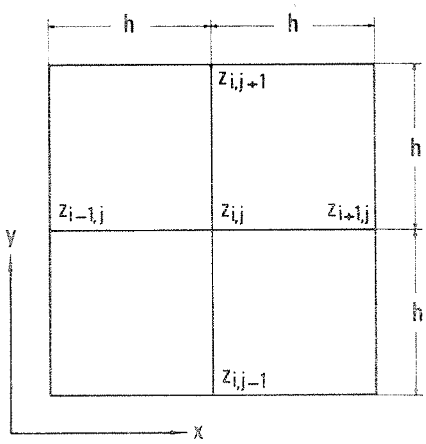

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but, since it is (2)

$$q = k \frac{H^2 - h^2}{2L} \tag{7}$$

we get:

$$g(x,0) = \frac{1}{2} H^2 - \frac{H^2 - h^2}{2L} x$$

To translate the relations (5) and (6) into variational inequality (a form particularly useful for the numerical approach) it is necessary to find the solution $w(x,y)$ of the problem into a particular set of functions; and precisely in the convex K that is a subset of the Sobolev's space $H'(R)$ of the functions defined in the rectangle R which have square summable with their first derivatives and furthermore which assume in this rectangle values > 0 and on the boundary ∂R the values of the function g seen before.

Indicating with $z(x,y)$ any function of this space, from Gauss Green's theorem, we get:

$$\int_R [(z - w)\, \Delta w + grad\, (z - w)\, grad\, w]\, dx\, dy = \int_{\partial R} (z - w) \frac{\partial w}{\partial x}\, dR$$

but, since $z \equiv w$ is on the boundary, we obtain:

$$\int_R grad\, (z - w)\, grad\, w\, dx\, dy = - \int_R (z - w)\, \Delta w\, dx\, dy$$

and remembering that:

$$\Delta w = 1 \quad \text{in } \Omega \quad \text{and} \quad \Delta w = 0 \quad \text{in } R - \Omega$$

$$\int_R grad\, (z - w)\, grad\, w\, dx\, dy = - \int_\Omega (z - w)\, dx\, dy = \int_R (w - z)\, dx\, dy -$$

$$- \int_{R-\Omega} (w - z)\, dx\, dy = \int_R (w - z)\, dx\, dy + \int_{R-\Omega} z\, dx\, dy$$

and given that $z \geq 0$, we get:

$$\int_R grad\, w\, grad\, (z - w)\, dx\, dy \geq \int_R (w - z)\, dx\, dy; \quad \forall z \in K \tag{8}$$

this relation represents a variational inequality in the variable w, which, as was already well-known, admits one and only one solution; furthermore we have rigorous methods to get the numerical solution and also to study the stability and convergence properties, etc. [2] [3].

(2) The relation (7) which up to some twenty years ago was believed to be only approximate because it was obtained from the Dupuit-Forchheimer model, is instead exact, as it has been demonstrated by many scientists. It is also easy to demonstrate the relation (7) applicating the (2) to the whole boundary $BCEFDAB$.

It's also interesting to note that, for the symmetry properties of (8), the problem under study is the same than to find the minimum in the space K, of the functional:

$$J(z) = \frac{1}{2} \int_R |grad\, z|^2\, dx\, dy + \int_R z\, dx\, dy \tag{9}$$

So, we can write down this problem in such a form: find $w \in K$, so that:

$$J(w) \leq J(z) \qquad \forall z \in K \tag{9'}$$

2. To calculate the solution of the minimum problem, it is possible to use approximation schemes based either on the finite differences or on the finite elements methods.

Using, for exemple, the finite differences method corrisponding to the classical formula of 5 points for the Δ operator, we introduce in the plan (x,y) a grid that, to be more simple, we assume as square (that is $\delta x - \delta y = h$) (fig.5); indicating than, respectivaly, with R_h and ∂R_h the set of nodes inside R and those on the boundary and with $z_{i,j}$ the generic vector defined on all the nodes R_h and ∂R_h, the problem consists in finding the vector $w_{i,j}$ that realizes the minimum of the following functional:

$$J(z_{ij}) = \frac{1}{2} \sum_{(i,j)\in R_h} [(z_{i+1,j} - z_{i,j})^2 + (z_{i-j} - z_{i,j})^2 + + (z_{i,j+1} - z_{i,j})^2 + (z_{i,j-1} - z_{i,j})^2] + h^2 \sum_{(i,j)\in R_h} z_{i,j} \tag{10}$$

among all the vectors $z_{i,j}$ so that: $z_{i,j} \geq 0$ on R_h and $z_{i,j} = g_h(i,j)$ on ∂R (where g_h is a function of g discreetized).

We can demonstrate that [4] the preceding problem (10), for each h, has only one solution, which for $h \to 0$ converges to the solution of the problem (9').

For the calculation of $\{w_{i,j}\}$, we see that the problem presented in such a way, is reduced to a problem of minimum of a quadratic functional on a convex set of finite dimension, that is to a tipical problem of quadratic programming, of which we know several schemes of solution. Particularly simple and efficient is the following way of proceeding, which represents an extension of the well-known relaxation process used to solve the linear systems; we start from an arbitrary vector $\{w^{(0)}_{i,j}\}$ with

$\{w^{(0)}_{i,j}\} = 0$ on R_h and $\{w^{(0)}_{i,j}\} = g_h(i,j)$ on ∂R_h and we

STREAMLINES

41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	2060	1465	715	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	4510	4145	3440	2469	1280	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	7241	6958	6347	5456	4323	2982	1469	0	0	0	0	0	0	0	0	0	0	0	0	0
33	10188	9947	9407	8599	7550	6281	4805	3139	1359	0	0	0	0	0	0	0	0	0	0	0
31	13313	13100	12614	11475	10901	9707	8301	6687	4860	2800	410	0	0	0	0	0	0	0	0	0
29	16597	16404	15961	15280	14373	13249	11914	10367	8603	6618	4400	1937	0	0	0	0	0	0	0	0
27	20022	19846	19440	18810	17964	16908	15643	14168	12476	10560	8407	5995	3342	240	0	0	0	0	0	0
25	23576	23415	23041	22458	21670	20680	19488	18088	16473	14632	12550	10204	7576	4604	1350	0	0	0	0	0
23	27249	27101	26757	26219	25488	24564	23446	22126	20594	18837	16836	14567	11999	9098	5854	2201	0	0	0	0
21	31030	30895	30580	30085	29411	28556	27515	26279	24839	23176	21270	19092	16608	13777	10558	6880	2686	0	0	0
19	34910	34788	34501	34051	33434	32649	31690	30546	29205	27649	25854	23786	21408	18669	15511	11859	7647	2844	0	0
17	38881	38771	38513	38107	37550	36839	35966	34921	33690	32254	30586	28651	26405	23791	20735	17150	12936	7992	2125	0
15	42932	42834	42606	42246	41751	41117	40336	39398	38288	36985	35464	33686	31604	29154	26251	22787	18625	13598	7529	590
13	47055	46970	46772	46460	46029	45476	44792	43969	42990	41837	40482	38888	37005	34764	32074	28806	24788	19786	13498	5605
11	51239	51168	51002	50738	50374	49905	49326	48625	47789	46800	45631	44248	42601	40621	38212	35235	31490	26678	20356	11830
9	55476	55419	55284	55071	54776	54396	53925	53354	52672	51860	50898	49752	48378	46711	44659	42082	38770	34389	28399	19916
7	59755	59712	59610	59449	59225	58937	58580	58145	57624	57004	56265	55381	54315	53011	51390	49328	46628	42967	37788	30074
5	64066	64037	63968	63860	63710	63517	63276	62983	62632	62212	61711	61109	60380	59484	58361	56917	55002	52358	48529	42577
3	68398	68383	68349	68295	68219	68122	68001	67854	67677	67465	67212	66907	66537	66080	65505	64761	63766	62380	60365	57420
1	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740	72740
	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39

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build a succession of vectors $\left\{\omega_{i,j}^{(n)}\right\}$; the passage from $\left\{\omega_{i,j}^{(n)}\right\}$ to $\left\{\omega_{i,j}^{(n+1)}\right\}$ is based on the following formulas:

$$\omega_{i,j}^{(n+1/2)} = \frac{1}{4}\left[\omega_{i-1,j}^{(n+1)} + \omega_{i+1,j}^{(n)} + \omega_{i,j-1}^{(n+1)} + \omega_{i,j+1}^{(n)} - h^2\right] \text{ on } R_h \qquad (11)$$

$$\omega_{i,j}^{(n+1)} = max\left[0, \omega_{i,j}^{(n)} + \omega(\omega_{i,j}^{(n+1/2)} - \omega_{i,j}^{(n)}\right] \qquad (12)$$

$$\omega_{i,j}^{(n+1)} = g_h(i,j) \quad \text{on} \quad \partial R_h \qquad (13)$$

where the expression (12) represents the projection of the vector $\left\{\omega_{i,j}^{(n+1)}\right\}$ on the convex; that is, indicating with:

$$a = \omega_{i,j}^{(n)} + \omega(\omega_{i,j}^{(n+1/2)} - \omega_{i,j}^{(n)})$$

we assume:

$$\omega_{i,j}^{(n+1)} = 0 \quad \text{if} \quad a < 0$$

and:

$$\omega_{i,j}^{(n+1)} = a \quad \text{if} \quad a \geq 0$$

We can demonstrate that such a numerical scheme converges, for each $\omega (0 < \omega < 2)$ where ω is a parameter that is used to accelerate the convergence.

It is interesting to note that at the end of the calcul in a part of the mesh there are values of ω equal to 0, and in the other part, the ω values are greater than 0; the free line is then obtained as an element of separation between those two zones (fig. 6).

It clearly results from this example that one of the advantages of the illustrated method consists in the fact that it makes it possible to operate within a known field (in our case that of the rectangle); for this it is possible to solve the problem without having to establish an approximate free line like, on the contrary is required with the usual ways of calculation, including the most recent ones, based, them too, on the varational principles.

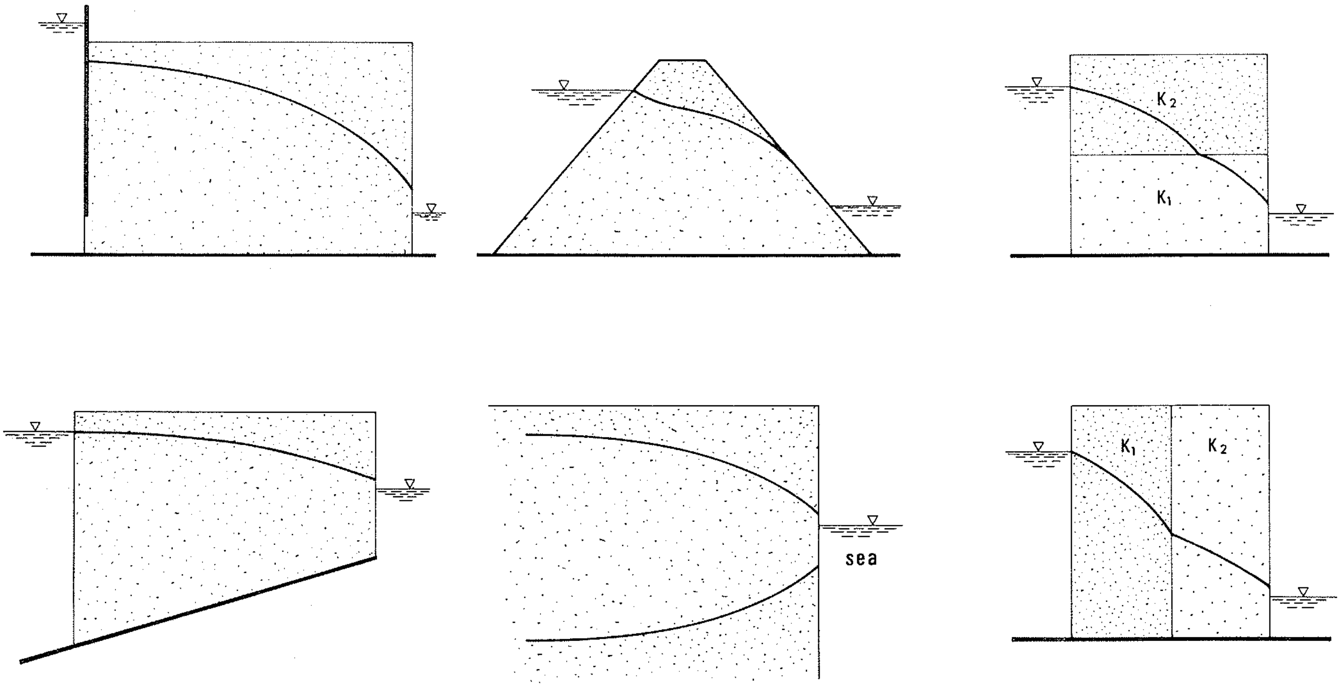
3. Like I said at the beginning, I willingly examined an extremely simple case of unconfined flow to make more simple the understanding of the basic principles of this method.

Within the field of these researches that are being done at the University of Pavia, we have solved a few others problems a little more interesting and complex like those illustrated in the fig. 7.

Their study was naturally more complex, and so, cannot be illustrated within such a short period of time: because of this I would advise those of you that are interested in this problem, to go back to the bibliography [1] [4] [5].

At the present stage of its elaboration, the illustrated method have not be generalized yet and for this it isn't applicable to any kind of unconfined flow.

Nevertheless, it has already appeared to be very versatile, and I think that, with the pregressing of the theoretical research, the method will be bettered until it can be applied to types of problems always more complicate and of major practical interest.



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Discussion

Présidents : MM. G. KOVACS et P. HABIB

M. le Président HABIB félicite M. le Professeur MAIONE pour son brillant exposé et ouvre la discussion.

M. le Professeur THIRRIOT intervient en ces termes :

Il y a beaucoup d'excellentes choses dans l'exposé du Professeur MAIONE et peut-être trop pour que, vu l'heure tardive, de trop nombreuses questions lui soient posées, mais j'ai éprouvé un plaisir particulier à voir que sa méthode répondait à une question que je me pose depuis longtemps, à savoir : comment utiliser le calcul variationnel ou relationnel pour déterminer les surfaces libres au moins dans leur principe.

En définitive, la fonctionnelle citée par le Professeur MAIONE à la fin de son exposé pourrait être concrétisée, d'une manière physique, comme représentant la somme de l'énergie dissipée en frottement et de l'énergie potentielle due à l'altitude dans le rectangle élémentaire considéré. Il est fort intéressant de voir que les espaces de Soboleff peuvent répondre à des questions aussi prosaïques.

M. le Professeur MAIONE répond :

Je suis vraiment heureux que le Professeur THIRRIOT ait trouvé tant de choses intéressantes dans le travail que j'ai présenté et, en particulier, la réponse à une question que depuis longtemps il s'était posée : il s'agit de la façon d'utiliser les méthodes variationnelles pour arriver à la détermination de la surface libre d'une nappe souterraine.

Je dois dire, à ce propos, que justement ceci a été le premier but de la recherche conduite par le groupe d'étude sur les milieux poreux qui s'appuie sur « L'Istituti di Matematica e di Idraulica» de l'Université de Pavie.

Je voudrais encore une fois souligner que la méthode exposée a la particularité de ne pas requérir l'établissement d'une surface libre de « tentative » comme on doit le faire dans les méthodes usuelles d'étude, y-compris les méthodes les plus récentes d'impostation variationnelle (ainsi que quelques unes des estimables notes présentées à cette même réunion l'ont montré).

On en retire un avantage qui n'est pas sans intérêt, tant pour son influence sur les temps de calcul que pour l'examen de questions théoriques de base telles que la stabilité, la convergence, la majoration de l'erreur, etc.

Je dois toutefois dire que la méthode n'est pas aussi facile lorsqu'on l'applique à l'examen de problèmes un peu plus compliqués comme celui d'un barrage en terre de profil trapézoïdal. En ce cas, en effet, on doit déterminer par tâtonnement trois paramètres : le débit, le point où la surface de la nappe apparaît à l'extérieur et la valeur de la variable w en un point de la base imperméable du barrage.

M. le Président HABIB remercie le conférencier et M. le Professeur THIRRIOT ; M. le Président KOVACS résume, comme suit, les conclusions de la séance :

Mesdames, Messieurs,

A mon avis, le programme de cet après-midi était très riche et très intéressant. Les discussions ont été si vivantes et si animées qu'il ne me reste guère de temps pour dégager une conclusion ou faire un résumé de tout ce qui a été dit..

Je crois pouvoir dire que je serais très tenté d'engager ou de poursuivre la discussion avec le Professeur MAIONE mais j'y renonce en raison de l'heure. J'espère néanmoins que nous pourrons reprendre cette discussion car les aperçus qu'il a donnés vont très loin.

Nous avons entendu des conférences extrêmement intéressantes proposant différentes méthodes, numériques et analogiques, pour l'utilisation rationnelle des ordinateurs. A ce propos, j'aimerais insister sur le point suivant : certains phénomènes ne peuvent pas être réduits à des modèles ou du moins aux modèles que nous utilisons actuellement ; je pense qu'il faut multiplier les recherches et les investigations. Si nous étudions par exemple un champ limité, il faudrait pouvoir ne négliger aucun phénomène physique, car en négliger un seul risque, dans certains cas, de modifier considérablement le résultat. En tout cas, il faut vérifier très attentivement les résultats physiques.

Je n'ai fait qu'un très bref résumé de la réunion, et je m'en excuse.