MOBILE BED FLUVIAL MATHEMATICAL MODELS

by

J. A. CUNGE

and

N. PERDREAU

Engineers at SOGREAH
(Société Grenobloise d'Etudes et d'Applications Hydrauliques)
Grenoble (France)

Nous reprenons, pour cet article, la présentation inaugurée dans le N° 1-1972, qui permet une publication rapide d'articles présentant une partie mathématique importante.

LA HOUILLE BLANCHE

MOBILE BED FLUVIAL MATHEMATICAL MODELS

By Jean A. Cunge[1] and Noël Perdreau[1]

INTRODUCTION

The mathematical model described in this paper may be used to solve numerous problems associated with the evolution of alluvial riverbeds. Such evolution may result from natural conditions or from structural construction work development.

- Some natural causes :
  . considerable bed variations in the lowest reach of rivers flowing into the sea which occur during the succession of flood and normal flow conditions, which results in accentuated rises of water level during rapid floods (5).
  . bed variations during floods upstream, downstream and through gorges (10).
  . variations downstream of tributaries due to irregular solide contribution.

- Some artificial causes :
  . modifications to liquid and solid hydrologic regimen resulting from dam construction. The longitudinal profile of the bed may be subject to marked long-term evolution.
  . modification of bed cross-sections due to river training.
  . slope modifications due to cut-offs or alignment, to the suppression or creation of localised head losses by the removal or construction of weirs, etc.
  . deposition upstream of dams which varies with reservoir level.
  . artificial modifications resulting from liquid and solid flow division at a branch.
  . dredging.

The mathematical model described in this paper may be used to determine the evolution of the longitudinal profile of a river, assuming that the form of the alluvial bed cross sections remain

1. Engineer at SOGREAH (Société Grenobloise d'Etudes et d'Applications Hydrauliques), Grenoble, France.

unchanged or that their evolution may be predetermined.

It is also supposed, as occurs in reality, that bed quilibrium is only affected by bed load material and, but insignificantly by suspended material. In the numerous problems mentioned above these assumptions are both justified and sufficient since only water and bed level results are required.

- Therefore the mathematical model offers a great economic saving as compared with the corresponding physical scale model.

A series of publications in which the authors have undertaken pioneer work in proving the feasibility of using such models (1, 2, 3, 4) are known. This paper describes the problems associated with one-dimensional mathematical models of mobile bed evolution. The writers describe a model adapted to simulate slow phenomena associated with solid transport in non-stationary flow. This model uses a stable economic numerical method so that computational time intervals compatible with the long periods studied may be adopted. Comprehensive study of the adopted implicit scheme of finite differences allows appreciation of approximation errors so that parameters (space and time intervals) for model discretisation may be correctly selected.

GOVERNING EQUATIONS OF THE ONE-DIMENSIONAL PROBLEM

Bed load transport and flow in a water course may be expressed by a set of three differential equations (see fig. 1) :

. Continuity equation of liquid flow

$$b\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \qquad (1)$$

. Dynamic equation of liquid flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} + K^2 u|u| = 0 \qquad (2)$$

. Continuity equation of solid discharge

$$\tilde{b}\frac{\partial z}{\partial t} + \frac{\partial G}{\partial x} = 0 \qquad (3)$$

together with the bed-load law

$$G = G(u, h) \qquad (4)$$

where x is the abscissa measured over the length of the river axis and expressed in meters ; t being time, expressed in seconds ; h being the depth of water, expressed in meters ; u the average velocity within a section, measured in meters per second ; z the level of the river bed, measured in meters above a reference level ; K the overall resistance coefficient ; G the solid load transported, in cubic meters per second ; b the width of the free surface, in meters ; $\tilde{b}$ is the width of the section affected by the bed load transport, in meters.

The set of differential equations 1, 2 and 3 links the three unknown functions $h(x,t)$, $u(x,t)$, $z(x,t)$ with the independent variables x and t.

Equations 1, 2 and 3 are the result of a series of hypothesis concerning both water flow and solid transport. The validity of Saint Venant's hypotheses is accepted for the liquid phase i.e. the hydrostatic distribution of pressure and the uniformity of velocity within a section. The phenomena of erosion and material deposition are of a three-dimensional nature due to secondary currents. It does not seem possible to reproduce this situation by calculation at least not over long reaches

of river whence its schematisation by a one-dimensional model. The model is based on Eqs. 1, 2, 3 and 4 defining the rising and lowering of the cross-section. It is accepted that cross-sections are displaced vertically within their own alluvia which is supposed as being vertically homogeneous. Thus initial cross-section form is invariable. Any other hypothesis would require the introduction of a law to cover the separation of eroded or deposited materials and a further law to cover their transversal division over the bed. The whole of the adopted hypotheses obviously excludes study of horizontal deformation of river beds. The choice of the set of equations described above may be criticized on the basis of interaction between the solid and liquid phases and it might be asked why a different set of formulae was not selected since this would in theory, more accurately represent the relevant phenomena. Representation of the two solid phases might be attempted : one phase would involve bed load transportation, the other suspension. It might also be held that the density of a water-material mixture varies with material concentration, etc.

Disregarding the consequent difficulties, such complications would not be justified in view of the basic hypotheses adopted for material movement and cross-section variations.

Thus the model based on Eqs. 1 to 4 represents only changes of the longitudinal bed profile as a function of time and hydraulic regimen.

MATHEMATICAL PROBLEM

Complete set of equations. The supposition (Eq. 4) that solid discharge G depends upon velocity u and depth h which permits replacement of the term $\partial G/\partial x$ in Eq. 3 by :

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial h}\frac{\partial h}{\partial x} \qquad (5)$$

Let $b = \tilde{b} = 1$ meter in Eqs. 1, 2 and 3 ; by adding to the equations so obtained, expressions for the total derivatives of functions u, h, and z a set of six equations is obtained :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} + K^2 u|u| = 0 \qquad (6)$$

$$\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} = 0 \qquad (7)$$

$$\frac{\partial z}{\partial t} + \frac{\partial G}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial G}{\partial h}\cdot\frac{\partial h}{\partial x} = 0 \qquad (8)$$

$$\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx - du = 0 \qquad (9)$$

$$\frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx - dh = 0 \qquad (10)$$

$$\frac{\partial z}{\partial t}\,dt + \frac{\partial z}{\partial x}\,dx - dz = 0 \qquad (11)$$

The condition which determines the characteristic directions of the set of Eqs. 6 to 8 is that the determinant of the set of Eqs. 6 to 11 is zero :

$$\begin{vmatrix} 1 & u & 0 & g & 0 & g \\ 0 & h & 1 & u & 0 & 0 \\ 0 & \partial G/\partial u & 0 & \partial G/\partial h & 1 & 0 \\ dt & dx & 0 & 0 & 0 & 0 \\ 0 & 0 & dt & dx & 0 & 0 \\ 0 & 0 & 0 & 0 & dt & dx \end{vmatrix} = 0 \qquad (12)$$

hence

$$-(dx)^3 + 2u(dx)^2\,dt + \left(gh - u^2 + g\frac{\partial G}{\partial u}\right) dx(dt)^2 + g\left(h\frac{\partial G}{\partial h} - u\frac{\partial G}{\partial u}\right)(dt)^3 = 0 \qquad (13)$$

Let $dx/dt = c$

$$-c^3 + 2uc^2 + \left(gh - u^2 + g\frac{\partial G}{\partial u}\right) c + g\left(h\frac{\partial G}{\partial h} - u\frac{\partial G}{\partial u}\right) = 0 \qquad (14)$$

The Eq. 14 and its analysis have been presented by de Vries (2) and Kyozo Suga (3). From their results it is noted that the set of Eqs. 6 to 8 has three characteristics in the plane (x, t).

Consequently each point of the domain, including the boundaries, is an intersection of the three characteristics curves (fig. 2). In subcritical flow conditions ($u^2 < gh$) a characteristic c_2 originating within the domain and two characteristics c_1 and c_3 originating outside the domain will pass through each upstream boundary point. Characteristic c_2, originating outside the domain, and two characteristics c_1 and c_3, originating within the domain, will pass through each downstream boundary point. Consequently, if the complete mixed problem of subcritical flow

is to be well posed from the mathematical standpoint it is necessary to give :

Three functions $u(x,0)$, $h(x,o)$ and $z(x,o)$ over the interval $0 \leqslant x \leqslant L$ at the initial instant $t = 0$.

Two conditions, functions of time, at the upstream boundary $x = 0$ of the model.

A time function condition at the downstream boundary $x = L$ of the model.

Simplified system. In view of the slow nature of the phenomena examined, liquid discharge $Q = u.h$ may be considered constant in time over the river reach. Thus a simplification of the previous case is obtained since $\partial(uh)/\partial x = 0$ and from Eq. 7 it is found that $\partial h/\partial t = 0$ and that $\partial u/\partial x = -(u/h)\cdot(\partial h/\partial x)$. By stating that $h = y - z$ and by substituting these results in Eq. 6 a set of two equations is obtained :

$$(1 - F^2)\,.\,\frac{\partial y}{\partial x} + F^2\frac{\partial z}{\partial x} + \frac{K^2}{g}\,u|u| = 0 \qquad (15)$$

$$\left(\frac{\partial G}{\partial h} - \frac{u}{h}\frac{\partial G}{\partial u}\right)\cdot\frac{\partial y}{\partial x} + \frac{\partial z}{\partial t} - \left(\frac{\partial G}{\partial h} - \frac{u}{h}\frac{\partial G}{\partial u}\right)\frac{\partial z}{\partial x} = 0 \qquad (16)$$

where $F = u/(gh)^{1/2}$ being the Froude number and y being the water line level. In this system the unknown functions are $y(x,t)$ and $z(x,t)$. By adding the expressions for the total derivatives of these functions to Eqs. 15 and 16 ;

$$\frac{\partial y}{\partial t}\,dt + \frac{\partial y}{\partial x}\,dx - dy = 0 \qquad (17)$$

$$\frac{\partial z}{\partial t}\,dt + \frac{\partial z}{\partial x}\,dx - dz = 0 \qquad (18)$$

and by equalizing at zero the determinant of the set of Eqs. 15 to 18

$$\begin{vmatrix} 0 & (1 - F^2) & 0 & F^2 \\ 0 & \left(\frac{\partial G}{\partial h} - \frac{u}{h}\frac{\partial G}{\partial u}\right) & 1 & -\left(\frac{\partial G}{\partial h} - \frac{u}{h}\frac{\partial G}{\partial u}\right) \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix} = 0 \qquad (19)$$

the equation for the characteristic directions $c = dx/dt$ of set 15, 16 may be obtained :

$$(1 - F^2)\, dx\, dt + \left(\frac{\partial G}{\partial h} - \frac{u}{h}\frac{\partial G}{\partial u}\right) dt^2 = 0 \qquad (20)$$

Hence the first solution corresponding to $dt = 0$ may be obtained i.e.

$$c_1 = \left(\frac{dx}{dt}\right)_1 = \infty \qquad (21)$$

Eq. 21 expresses that the characteristic c_1 is parallel to the x axis i.e. that the velocity of the short liquid waves is infinite. Eq. 20 also provides the second characteristic direction

$$c_3 = \left(\frac{dx}{dt}\right)_3 = \frac{u \frac{\partial G}{\partial u} - h \frac{\partial G}{\partial h}}{h\,(1 - F^2)} \qquad (22)$$

The value c_3 expressed in meters per second represents the velocity of the small bed waves and is identical to that obtained by de Vries (2) if one ignores the dependence of solid flow on depth h. The simplified set of Eqs. 15 and 16 differs from the complete set 6, 7, 8. Each point of the domain considered in the plane (x, t) is a point of intersection of two curves : a straight line c_1 parallel to the abscissa x and a characteristic c_3 defined by Eq. 22 (Fig. 3). To ensure that the problem is well posed, the following is necessary :

Two functions $y(x,o)$ and $z(x,o)$ over the interval : $o \leqslant x \leqslant L$ at the initial moment $t = 0$;

A time function condition at the upstream limit $x = 0$ on the model ;

A time function condition at the downstream limit $x = L$ of the model.

It may be shown that in the typical case of large rivers $c_3 \ll (c_1, c_2)$ i.e. that the bed load transport phenomena are slow compared with liquid wave propagation (2), (3). Equation 22 shows that this hypothesis is only valid for flow where the Froude number differs considerably from 1.

GOVERNING EQUATIONS OF THE PROBLEM OF CONSTANT DISCHARGE CASE

It has already been stated that the hypothesis of constant liquid discharge allows the set of three equations, 6 to 8, to be reduced to two equations :

. energy line equation

$$\frac{\partial}{\partial x}\left(\frac{Q^2}{2S^2} + gy\right) + \frac{g}{D^2} Q|Q| = 0 \qquad (23)$$

. solid flow continuity equation

$$\frac{\partial z}{\partial t} + \frac{1}{\tilde{b}} \frac{\partial G}{\partial x} = 0 \qquad (24)$$

where Q is constant and represents the liquid discharge, expressed in cubic meters per second ; S is the wetted surface expressed in square meters ; $\tilde{b}$ is the width being affected by bed load transport, expressed in meters ; D is the conveyance of the section ; y is the free surface level expressed in meters above a reference level (see Fig. 1).

The set of equations 23 and 24 forms a system of non-linear hyperbolic partial differential equations From this system solutions i.e. functions $y(x,t)$ and $z(x,t)$ may be obtained over a strip $0 \leqslant x \leqslant L$, $0 \leqslant t \leqslant T$ of the plane (x,t) if the problem is well posed. It should be noted that this is not a classical Cauchy problem since one of the characteristic directions of the system is parallel to the x axis, i.e. initial values are given along the characteristic.

The numerical solution of the set 23, 24 was proposed by M. de Vries (ref. 2, 3) by an explicit method of finite differences. Although this method seems satisfactory for a great number of cases, it requires, as is the case for any explicit scheme, a limit to the calculation time interval Δt. A method based on an implicit finite difference scheme is presented to overcome this limitation. The method is unconditionally stable and allows arbitrary time intervals to be adopted while avoiding numerical instability.

DISCRETISATION OF NON LINEAR DIFFERENTIAL SYSTEM 23, 24

Any function $f(x,t)$ and its derivatives $\partial f/\partial x$ and $\partial f/\partial t$ are replaced in Eqs. 23 and 24 by the following approximations :

$$f(x,t) \sim \theta \cdot \frac{f_j^{n+1} + f_{j+1}^{n+1}}{2} + (1-\theta) \cdot \frac{f_j^{n} + f_{j+1}^{n}}{2} ;$$

$$\frac{\partial f}{\partial x} \sim \theta \frac{f_{j+1}^{n+1} - f_j^{n+1}}{\Delta x} + (1-\theta) \cdot \frac{f_{j+1}^{n} - f_j^{n}}{\Delta x} ; \qquad (25)$$

$$\frac{\partial f}{\partial t} \sim \frac{1}{2} \left\{ \frac{f_{j+1}^{n+1} - f_{j+1}^{n}}{\Delta t} + \frac{f_j^{n+1} - f_j^{n}}{\Delta t} \right\} ;$$

whence $f_j^n = f(j.\Delta x, n.\Delta t)$; $f_{j+1}^n = f[(j+1)\,\Delta x, n\,\Delta t]$ etc.

and $0 \leqslant \theta \leqslant 1$ being a weighting coefficient.

This may be written as

$$f_j^{n+1} = f_j + \Delta f_j \; ; \; (f_j^n = f_j) \; ; \qquad (26)$$

where Δf_j is the unknown increment of the f function between times $t = n.\Delta t$ and $(n+1).\Delta t$ at point j.

The substitution of Eqs. 25 and 26 in Eq. 23 gives :

$$\frac{\theta}{\Delta x} \left\{ \frac{Q^2}{2(S_{j+1} + \Delta S_{j+1})^2} + g(y_{j+1} + \Delta y_{j+1}) - \frac{Q^2}{2(S_j + \Delta S_j)^2} - g(y_j + \Delta y_j) \right\} +$$

$$+ \frac{1-\theta}{\Delta x} \left\{ \frac{Q^2}{2S_{j+1}^2} + gy_{j+1} - \frac{Q^2}{2S_j^2} - gy_j \right\} +$$

$$+ gQ|Q| \cdot \left\{ \frac{\theta}{2} \left[\frac{1}{(D_{j+1} + \Delta D_{j+1})^2} + \frac{1}{(D_j + \Delta D_j)^2} \right] + \frac{1-\theta}{2} \left[\frac{1}{D_{j+1}^2} + \frac{1}{D_j^2} \right] \right\} = 0 \qquad (27)$$

Also a finite difference equivalent may be obtained for Eq. 24 :

$$\frac{\tilde{b}_{j+1} \cdot \Delta z_{j+1} + \tilde{b}_j \, \Delta z_j}{2\Delta t} + \frac{\theta}{\Delta x}(G_{j+1} + \Delta G_{j+1} - G_j - \Delta G_j) + \frac{1-\theta}{\Delta x}(G_{j+1} - G_j) = 0 \qquad (28)$$

566

Increments of the wetted area S, conveyance D and solid discharge G may be expressed by increments of the unknown functions y and z. Since

$$S = S(h) = S(y-z) \; ; \quad D = D(y-z) \quad ; \quad G = G(y-z)$$

then :

$$\Delta S_j = \frac{dS_j}{dh}(\Delta y_j - \Delta z_j) = b_j\,(\Delta y_j - \Delta z_j)$$

$$\Delta D_j = \frac{dD_j}{dh}(\Delta y_j - \Delta z_j) = D_j'\,(\Delta y_j - \Delta z_j) \qquad (29)$$

$$\Delta G_j = \frac{dG_j}{dh}(\Delta y_j - \Delta z_j) = G_j'\,(\Delta y_j - \Delta z_j)$$

where b_j is the width of the free surface corresponding to depth h at point j, expressed in meters.

The substitution of 29 in Eqs. 27 and 28 results in a set of non-linear algebraic equations with unknows Δy and Δz. However, these are only small variations during a time interval : Δt, and in the great majority of cases the equations may be linearised. This is done by developing the Eqs. 27 and 28 (in which 29 is substituted) into Taylor series in terms of Δy and Δz and by truncating the higher order terms. The result is, for each pair of points, (j,j+1) a system of two linear algebraic equations linking the unknows Δy_j, Δz_j, Δy_{j+1} and Δz_{j+1} :

$$A_1\,\Delta y_j + B_1\,\Delta z_j + C_1\,\Delta y_{j+1} + D_1\,\Delta z_{j+1} + H_1 = 0$$

$$A_2\,\Delta y_j + B_2\,\Delta z_j + C_2\,\Delta y_{j+1} + D_2\,\Delta z_{j+1} + H_2 = 0 \qquad (30)$$

where A_1, B_1, ... etc. are functions of values known at time t, at points j and j+1. The mathematical model of a reach contains N computation points and there are 2(N-1) type 30 equations. As was seen earlier, the system is only well posed if there is at each moment one known condition at each boundary. Thus there are 2(N-1)+2 = 2N equations for finding 2N unknowns Δy_j, Δz_j (j = 1, 2, ... N) for determining the levels of the river bed z_j^{n+1} and the free surface y_j^{n+1} at the time $(n+1).\Delta t$.

SOLUTION OF ALGEBRAIC SYSTEM

Solution of the set of 2N linear equations (Eq. 30) is ensured by the double sweep method (9) described briefly here.

Assuming a set of Eqs. 30 for N computation points. If a linear relationship can be established for any point j (this is a hypothesis) :

$$\Delta z'_j = E_j \, \Delta y_j + F_j \qquad (31)$$

then a relationship of the same type may be obtained for point j + 1 :

$$\Delta z_{j+1} = E_{j+1} \, \Delta y_{j+1} + F_{j+1} \qquad (32)$$

where E_j, F_j, E_{j+1} and F_{j+1} are coefficients. This proposition may be proved by substitution of Eq. 31 into Eqs. 30, thus eliminating Δz_j and by subsequently eliminating Δy_j from two equations. Thus a relationship between Δz_{j+1} and Δy_{j+1} is found :

$$\Delta z_{j+1} = - \frac{C_1 (A_2 + B_2 E_j) - C_2 (A_1 + B_1 E_j)}{D_1 (A_2 + B_2 E_j) - D_2 (A_1 + B_1 E_j)} \cdot \Delta y_{j+1} - \frac{(H_1 + B_1 E_j)(A_2 + B_2 E_j) - (H_2 + B_2 E_j)(A_1 + B_1 E_j)}{D_1 (A_2 + B_2 E_j) - D_2 (A_1 + B_1 E_j)} \qquad (33)$$

Eq. 33 is similar to Eq. 32 with $E_{j+1} = E_{j+1}(E_j)$ and $F_{j+1} = F_{j+1}(E_j, F_j)$. Consequently, if the coefficients E_1 and F_1 are known for the boundary condition x = 0 then, using Eq. 26 one can compute by recurrence the coefficients E_j, F_j for all points j = 2, 3,..... N-1, N. Replacing Δz_j in Eqs. 30 by relationship 31 and eliminating Δz_{j+1} from the two remaining equations, one obtains :

$$\Delta y_j = \frac{C_1 D_2 - C_2 D_1}{D_1 (A_2 + B_2 E_j) - D_2 (A_1 + B_1 E_j)} \cdot \Delta y_{j+1} - \frac{(H_2 D_1 - H_1 D_2) + (B_2 D_1 - B_1 D_2) F_j}{D_1 (A_2 + B_2 E_j) - D_2 (A_1 + B_1 E_j)} \qquad (34)$$

or again :

$$\Delta y_j = L_j \, \Delta y_{j+1} + M_j \qquad (35)$$

Thus Δy_{j+1} being known one can obtain Δy_j by Eq. 35 and Δz_j by Eq. 31.

The two Eqs. 31 and 35 define the method of solving the 2(N-1) type 30 algebraic equations, completed by two boundary conditions. Based on a known state $(z_j^n, y_j^n ; j = 1, 2, \ldots N)$ at time $t = n.\Delta t$, the condition at a boundary, for example x = 0, may be expressed in the form $\Delta z_1 = E_1 \Delta y_1 + F_1$. Subsequently using Eq. 33, the coefficients E_j, F_j may be calculated for all points j = 2,3.... N. Conditions at the other boundary x = L are used to define Δy_N, and Eq. 31 for finding $\Delta z_N = E_N \Delta y_N + F_N$. $y_N^{n+1} = y_N^n + \Delta y_N$, $z_N^{n+1} = z_N^n + \Delta z_N$, are then calculated using formulae 35 and 31. Thus Δy_j, Δz_j and, subsequently, y_j^{n+1}, z_j^{n+1} at all points j = N-1, N-2, ..., 2, 1 ; may be determined.

There is no need to dwell upon the advantages of this method which requires a number of arithmetical operations only proportional to the number of points N as compared with the inversion of the matrix requiring a number of operations proportional to N^3.

APPLICATION OF THE MODEL TO THE NON-LINEAR CASE

In order to test the mathematical model, which will be known as MODEL II, it would be desirable to use it to reproduce the evolution of a sufficiently long river whose behaviour has been observed with sufficient accuracy over an adequate period.

If such data were unavailable then the mathematical model could be used to reproduce long-term evolution of a mobile bed laboratory flume. The adjustment of the bed slope over a sufficiently long reach of the flume against the liquid and solid discharge introduced from upstream would be observed. Such observations would involve large-scale construction work and tests of considerable duration and consequent heavy expenditure.

The construction of such a flume was deemed unnecessary since satisfactory comparison had already been effected between the evolution of the londitudinal profile of a physical scale mobile bed model and a mathematical model (5). But this case, i.e. the rapid evolution of a fluvial bed, was a priori, considered to be more difficult to represent by calculation. This comparison was undertaken using a mathematical model, which will be termed MODEL I, to reproduce transient phenomena of the two phases (liquid flow and solid discharge) variable in time and space. The detailed description of this model, based upon the method with a limited computation time interval is given in reference (5).

The writers considered it sufficient to test MODEL II by comparing its results with those given by MODEL I which was considered as a reference standard.

So two models, I and II, were built up, reproducing a reach of river with the following characteristics : length 10 kilometers, rectangular section, constant width : b = 100 meters, initial bed slope i_o=0.004, grain diameter d = 35 millimeters, specific weight of material 2.68 tons per cubic meter, porosity 1.0, overall roughness (Strickler coefficient) 30. At the outset liquid discharge of Q = 1 187 cubic meters per second was assumed, with a normal depth h = 3 meters. The following conditions were given at the reach boundaries : downstream the bed level is z = constant, upstream the solid transport load varies versus time as per Table 1. The validity of Meyer-Peter's formula (6) was accepted for these two models :

$$G = (R.\ h.\ J - T)^{3/2} \qquad (36)$$

where h is the depth in meters, $J = Q^2/D^2$ is the slope of the energy line, R and T are numerical coefficients depending upon the characteristics of the materials transported. In the example dealt with here, R = 96.4, T = 0.36. From Eq. 36, the transport corresponding to the initial state was found to be 0.718 cu m per sec while the increase in solid discharge, defined by Table 1, causes the bed slope to tend towards a limit of 0.005. The two models both contain the same number of computation point : 101 i.e. 100 basic intervals of Δx = 100 m. The discretisation of the MODEL II boundary conditions for this example is given in APPENDIX I. The calculations were intended to test the validity of MODEL II (by varying discretisation parameters, i.e. time step Δt and the weighting coefficient θ) compared with MODEL I. The list of runs undertaken and their enumeration is given in Table 2.

Calculation results are shown in Figs. 5, 6 and 7.

Fig. 5 shows the variation of bed level at the upstream boundary as a function of time. Fig. 6 shows the longitudinal profile of river bed at time t = 116 hours. Fig. 7 shows the river bed longitudinal profile at various times for run N° 3.

I should be noted that :

- Runs N° 2 and N° 4 produce practically the same results as run N° 1 which served as a base reference.

- Computation is stable for θ = 0.5 and θ = 1.0 ; this result will be confirmed in the following chapter concerning numerical stability analysis of the adopted difference scheme. Value of the coefficient θ has no essential effect on the results provided that $\theta \geqslant 0.5$.

- Artificial attenuation of waves may be seen when the ratio $\Delta t/\Delta x$ increases but this is acceptable for slow variations, even when the ratio becomes considerable. This attenuation may be seen in Fig. 6 by comparing the results obtained for Δt = 4 hours and Δt = 100 hours.

- The longitudinal bed profiles respect continuity : the volume of material deposited is equal to the difference between the quantity of material at the upstream entry and the downstream outlet.

- MODEL II is better adapted for the simulation of long term phenomena then MODEL I since time intervals 400 times greater may be adopted without fundamentally affecting the results.

NUMERICAL STABILITY - CONVERGENCE

The numerical solution of a set of non-linear differential equations, such as Eqs. 23 and 24, raises question of numerical stability and convergence of the finite difference scheme. There is no existing theory for analysing non-linear equations problems. However, the analysis of small perturbation may be applied to the linearized form of differential equations and to their finite difference analogues. This analysis generally allows estimation of the domain of stability and of the convergence properties of the scheme applied to a linearized problem. Transposition of these conclusions to non-linear case must be checked by experimental calculations but, generally, the conditions defined for linear cases are confirmed in practice.

The ideas of Preissmann (7) and Leendertse (8) are followed in this chapter

Linearisation of differential system. The Eqs. 23, 24 are linearised around the uniform flow writing :

$$y = y_o + \eta \quad ; \quad z = z_o + \zeta \tag{37}$$

where the symbols with o subscript correspond to the uniform flow while η and ζ are small perturbations of independent variables. By substituting Eq. 37 in Eqs. 23, 24 and by stating $\partial f_o/\partial t = 0$, $\partial f_o/\partial x = 0$ for all o subscripted functions a linearized form of 23, 24 is obtained if the higher order terms are ignored :

$$\left.\begin{aligned} &\frac{\partial \zeta}{\partial t} + P_1 \frac{\partial \eta}{\partial x} - P_1 \frac{\partial \zeta}{\partial x} = 0 \\ &P_2 \frac{\partial \eta}{\partial x} + P_3 \frac{\partial \zeta}{\partial x} + P_4 \eta - P_4 \zeta = 0 \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &P_1 = \frac{1}{\tilde{b}_o}\,\frac{dG}{dh} = \frac{G'}{\tilde{b}_o} \;; \qquad P_2 = 1 - F_o^2 \;; \\ &P_3 = F_o^2 \;; \qquad P_4 = -\frac{2Q^2}{D_o^2}\,\frac{dD}{dh} = -\frac{2Q^2 D'}{D^2_o} \;; \end{aligned}\right\} \tag{39}$$

Differential system 38 can be solved exactly. Particularly, supposing that a sinusoidal wave represents the initial state, the solution of system 38, i.e. the wave transformed by it after a certain time t, may be found. This transformation is a solution of Eq. 38. The equivalent of Eqs. 38 may be expressed in finite differences using the implicit scheme of Eq. 25, suppose again a similar sinusoidal perturbation as an initial state (although defined only at grid points). If it is possible to solve the difference system after the same time lapse t, i.e. to find the wave transformed by the system of difference equations equivalent to the Eqs. 38, then one may compare the solutions of these two systems. As the systems are linear, the superposition of the solution is valid and even complex waves may be represented by Fourier series. During propagation each sinusoidal wave may be damped (or amplified) and, furthermore its celerity might vary. The resulting wave may thus be deformed by the damping and dispersion of its component waves.

Leendertse (8) has defined two coefficients of convergence based on the comparison of deformed waves :

$$R_1 = \frac{\text{wave damping by the finite difference system}}{\text{wave damping by the original system (38)}} \tag{40}$$

$$R_1 = \frac{\text{wave celerity in the finite difference system}}{\text{wave celerity in the system (38)}} \tag{41}$$

If the coefficients R_1 and R_2 are equal to 1 then the numerical solution obtained by the use of finite differences is the same as the exact analytical solution. If not then the value of these coefficients indicates a measure of convergence of the approximate solution towards the analytical solution.

Analytical solution of Eqs. 38. An analytical solution of the linear system 38 in sinusoidal wave form was sought :

$$\eta = \eta_o \exp\,[i\,(\beta t + \sigma x)] \;; \quad \zeta = \zeta_o \exp\,[i\,(\beta t + \sigma x)] \tag{42}$$

where $i = \sqrt{-1}$; $\sigma = 2\pi/L = 2\pi k$, corresponds to the wave number k ; $\beta = 2\pi/T = 2\pi\nu$, corresponds to wave frequency ν ; L is the wave length ; T is the wave period. Substitution of Eq. 42 in Eq. 38 provides the homogeneous system :

$$\left.\begin{aligned} P_1 \sigma \eta_o + (\beta - P_1 \sigma)\,\zeta_o &= 0 \\ (P_4 + iP_2\sigma)\,\eta_o + (-P_4 + iP_3\sigma)\,\zeta_o &= 0 \end{aligned}\right\} \tag{43}$$

the determinant of which must be zero if there are to be non-trivial solutions for η_o and ζ_o. From this condition of determinant nullity and by allowing for $P_2 + P_3 = 1$, is obtained :

$$\beta = \frac{P_1 P_2 \sigma^3}{P_4^2 + P_2^2 \sigma^2} + i \cdot \frac{P_1 P_4 \sigma^2}{P_4^2 + P_2^2 \sigma^2} \tag{44}$$

It can be seen that β is a complex number $\beta = \mathrm{Re}\beta + i\,.\,\mathrm{Im}\beta$. Given that $dG(h,u)/dh < 0$, the term $P_1 \,.\, P_4 > 0$ and consequently $\mathrm{Im}\beta > 0$. The damping factor of the sinusoidal solution of Eq. 38 is $\exp(-\mathrm{Im}\beta t)$ and for $\mathrm{Im}\beta > 0$ one finds $\exp(-\mathrm{Im}\beta t) < 1$ which implies that an initial perturbation of type 42 will always be damped with time. The celerity of such perturbation will be a function of σ :

$$c = \frac{L}{T} = \frac{\mathrm{Re}\beta}{\sigma} = \frac{P_1 P_2 \sigma^2}{P_4^2 + P_2^2 \sigma^2} \qquad (45)$$

this means that dispersion will occur since the celerity of each wave depends upon its number and thus upon its length.

It should be noted that damping and dispersion of perturbations 42 by system 38 result from term P_4 corresponding to friction. By stating $P_4 = 0$ one obtains :

$$\beta = \mathrm{Re}\beta = \frac{P_1}{P_2}\sigma \;; \qquad \mathrm{Im}\beta = 0 \;; \qquad (46)$$

whence $\exp(-\mathrm{Im}\beta t) = 1$; $c = P_1/P_2$; this means that in a frictionless system 38 a sinusoidal wave retains its initial form irrespective of its length.

Approximate solution of linear system. The numerical solution to the set of equation 38 was sought by means of an implicit finite difference scheme already used for the non-linear system 23, 24.

The replacement of the partial derivatives and functions in Eqs. 38 by the finite differences defined by Eqs. 25 gives :

$$\left.\begin{aligned}
&\frac{\zeta_{j+1}^{n+1} - \zeta_{j+1}^{n} + \zeta_{j}^{n+1} - \zeta_{j}^{n}}{2\Delta t} + \frac{P_1}{\Delta x}\left[\theta\,(\eta_{j+1}^{n+1} - \eta_{j}^{n+1}) + (1-\theta)(\eta_{j+1}^{n} - \eta_{j}^{n})\right] \\
&\qquad - \frac{P_1}{\Delta x}\left[\theta(\zeta_{j+1}^{n+1} - \zeta_{j}^{n+1}) + (1-\theta)\,(\zeta_{j+1}^{n} - \zeta_{j}^{n})\right] = 0 \;; \\
&\frac{P_2}{\Delta x}\left[\theta\,(\eta_{j+1}^{n+1} - \eta_{j}^{n+1}) + (1-\theta)(\eta_{j+1}^{n} - \eta_{j}^{n})\right] + \frac{P_2}{\Delta x}\left[\theta(\zeta_{j+1}^{n+1} - \zeta_{j}^{n+1}) + (1-\theta)(\zeta_{j+1}^{n} - \zeta_{j}^{n})\right] \\
&+ \frac{P_4}{2}\left\{\left[\theta(\eta_{j+1}^{n+1} + \eta_{j}^{n+1}) + (1-\theta)(\eta_{j+1}^{n} + \eta_{j}^{n})\right] - \left[\theta(\zeta_{j+1}^{n+1} + \zeta_{j}^{n+1}) + (1-\theta)(\zeta_{j+1}^{n} + \zeta_{j}^{n})\right]\right\} = 0 \;;
\end{aligned}\right\} \qquad (47)$$

Let the solution of the set 47 be :

$$\left.\begin{aligned}
&\eta_{j}^{n} = \eta_o.\exp\left[i(\beta n\Delta t + \sigma j\Delta x)\right] \;; \quad \zeta_{j}^{n} = \zeta_o \exp\left[i(\beta n\Delta t + \sigma j\Delta x)\right] \\
&\eta_{j+1}^{n} = \eta_o.\exp\left[i\,\{\beta n\Delta t + \sigma(j+1)\Delta x\}\right] \;; \quad \zeta_{j+1}^{n} = \zeta_o \exp\left[i\,\{\beta n\Delta t + \sigma(j+1)\Delta x\}\right] \\
&\eta_{j}^{n+1} = \eta_o \exp\left[i\,\{\beta(n+1)\Delta t + \sigma j\Delta x\}\right] \;; \;\ldots \\
&\text{etc.}
\end{aligned}\right\} \qquad (48)$$

analogous for $t = n.\Delta t$ and $x = j.\Delta x$ in Eqs. 42.

By substituting the solutions 48 in Eqs. 47 and by dividing the two equations by $\exp[i(\beta n\Delta t + \sigma j\Delta x)]$, regrouping their terms and dividing again by $[\exp(i\sigma\Delta x)+1]$ and stating :

$$\left.\begin{aligned}
&\frac{\exp(i\sigma\Delta x) - 1}{\exp(i\sigma\Delta x) + 1} = i\,\tan(\sigma\Delta x/2) = i.\tan\alpha \;; \\
&\alpha = \frac{\sigma\Delta x}{2} \;;
\end{aligned}\right\} \qquad (49)$$

a system of two homogeneous equations is obtained :

$$\left.\begin{aligned}
&\zeta_o\{[\exp(i\beta\Delta t) - 1] - P_1 r.[\theta\exp(i\beta\Delta t) + 1 - \theta]\, i\tan\alpha\} + \\
&+ \eta_o\{P_1 r\,[\theta\exp(i\beta\Delta t) + 1 - \theta]\, i\tan\alpha\} = 0 \;; \\
&\zeta_o\left[i\frac{P_3}{\Delta x}\tan\alpha - \frac{P_4}{2}\right] + \eta_o\left[i\frac{P_2}{\Delta x}\tan\alpha + \frac{P_4}{2}\right] = 0 \;;
\end{aligned}\right\} \qquad (50)$$

whence $r = 2\Delta t/\Delta x$

The nullity condition of the determinant of the Eqs. 50 gives a solution for $\exp(i\beta\Delta t)$:

$$\exp(i\beta\Delta t) = 1 - \frac{P_1 r\tan^2\alpha.\left(\frac{P_4\Delta x}{2} + P_1 r\theta\tan^2\alpha - iP_2\tan\alpha\right)}{P_2^2\tan^2\alpha + \left(\frac{P_4\Delta x}{2} + P_1 r\theta\tan^2\alpha\right)^2} \qquad (51)$$

β being a complex number, it may be stated that $\beta = \mathrm{Re}\beta\Delta t + i\,\mathrm{Im}\beta\Delta t$ and to separate equation 51 by equalizing the real and imaginary parts.. This gives :

$$\exp(-\mathrm{Im}\beta\Delta t)\cos(\mathrm{Re}\beta\Delta t) = 1 - \frac{P_1 r \tan^2\alpha\left(P_1 r\theta\tan^2\alpha + \frac{P_4\Delta x}{2}\right)}{P_2^2\tan^2\alpha + \left(\frac{P_4\Delta x}{2} + P_1 r\theta\tan^2\alpha\right)^2} \qquad (52)$$

$$\exp(-\mathrm{Im}\beta\Delta t)\sin(\mathrm{Re}\beta\Delta t) = \frac{P_1\,P_2\,r\,\tan^3\alpha}{P_2^2\tan^2\alpha + \left(\frac{P_4\Delta x}{2} + P_1 r\theta\tan^2\alpha\right)^2} \qquad (53)$$

From Eqs. 52, 53 the damping factor $\exp(-\mathrm{Im}\beta\Delta t)$ and the celerity $(\mathrm{Re}\,\beta\Delta t/\sigma\Delta t)$ of the approximate solution may be found numerically as functions of $\sigma\Delta x$, θ, r, P_1, P_2, P_3, P_4.

The damping factor $\exp(-\mathrm{Im}\beta\Delta t)$ indicates the numerical stability of the finite difference scheme. If $\exp(-\mathrm{Im}\beta\Delta t) < 1$ the solution of the system is damped with time. Whereas if $\exp(-\mathrm{Im}\beta\Delta t) > 1$ this means that the initially small perturbations will increase with time, i.e. that the scheme is unstable.

Convergence coefficients. The sumerical solution of the Eqs. 52, 53 determines the zones of stability of the scheme and permits calculation of coefficients R_1 and R_2, defined by equations 40, 41. To eliminate a certain number of parameters, the coefficients P_1, P_2, P_3 and P_4 are initially defined by adopting the characteristics of a theoretical water course as described in the earlier chapter.

Thus for a uniform flow it is found that $h_o = 3.0$ m, $Q/b = 11.87$ cu m per sec, $P_1 = -0.01216$ m per s, $P_2 = 0.467$, $P_3 = 0.533$, $P_4 = -0.00445\ m^{-1}$.

Using these values and from Eqs. 52, 53, are found the curves giving the damping coefficient $\exp(-\mathrm{Im}\beta\Delta t)$ in terms of the weighting coefficient θ, of the ratio $r/2 = \Delta t/\Delta x$ and of ratio $L/\Delta x$. This latter expresses the extent of spatial discretisation (number of space intervals Δx/wave length L) or, conversely, the wave length expressed in Δx. These curves are shown in Fig. 8.

571

The curves of coefficient R_1 defined from Eq. 40 as a function of θ, $\Delta t/\Delta x$ and $L/\Delta x$ are shown in Fig. 9. Similarly, values of coefficient R_2 obtained from Eq. 41 are shown in Fig. 10.

Examination of the results presented permits the following comments on stability and convergence.

- The system is numerically stable for all tested values between $0.5 \leqslant \theta \leqslant 1.0$. The system is found to be unstable when $\theta < 0.5$ since $\exp(-\mathrm{Im}\beta\Delta t) > 1$. This is not formal proof that the stability condition is $0.5 < \theta < 1.0$ though it corroborates the results obtained for the non-linear system. It may be proved (see APPENDIX II) that this condition is necessary for stability in a frictionless case.

- Convergence of results depends upon the time step, selected as a function of space interval and upon the wave length. Thus from Fig. 9, when $\theta = 1.0$, a coefficient $R_1 \leqslant 1.10$ may be found (10 % amplitude error compared with the analytical solution) for wave length $L = 5\Delta x$ if $\Delta t = 10\Delta x$ and also for wave length $L = 80\Delta x$ if $\Delta t = 3000\Delta x$. Or consider a mathematical model with spatial interval Δx_1 simulating the propagation of the sinusoidal wave of length $L_1 = 80\Delta x_1$ using the time step $\Delta t_1 = 3000\Delta x_1$. If it is desired to calculate the propagation of a wave of length $L_2 = 5\Delta x_1$, using the same model, then to obtain the same accuracy of damping time step $\Delta t_2 = 10\Delta x_1$ must be adopted. It will be seen that parasite damping due to the finite difference approximation is particularly marked for short waves but has no great effect upon results for long perturbations. It should be emphasized that, in certain cases, damping may falsify results if the time step is too long but calculations are always stable for $\theta \geqslant 0.5$.

The same comments my be applied to coefficient R_2 giving the measure of acceleration or retardation of waves due to a finite difference system.

It will be seen from Fig. 8 to 10 that it is impossible to obtain $R_1 = 1$ or $R_2 = 1$ (i.e. coincidence of analytical and numerical solutions) for general case. In the frictionless system ($P_4 = 0$) it will be found that $R_1 = R_2 = 1$ if $\theta = 0.5$, $\Delta t = 0.5\,\Delta x$ and $P_1 = P_2$ (see APPENDIX II).

It should be noted that all these conclusions are drawn from an example based on determined values of coefficients P_1, P_2, P_3 and P_4. The curves in Figs. 8, 9 & 10 would be changed if the coefficients are varied. Nevertheless their shape would be generally unaltered.

CONCLUSION

The mathematical model presented in this paper may be used to simulate long term evolution of a river flowing over its own alluvia if the problem may be considered as one-dimensional and if discharge versus time variations are slow.

A method allowing the construction of mathematical models was presented and the theoretical and numerical aspects of problems were emphasised.

Naturally, in practice, mathematical models, like physical scale models, require adjustment using historical data on the evolution of the longitudinal profile of the river.

The use of mathematical models for this type of problem is more rapid, more flexible, less costly and less limited than that of physical scale models.

ACKNOWLEDGEMENTS

The writers wish to thank Messrs Blanchet C. and Preissmann A., Chief Engineers at SOGREAH for their advice.

APPENDIX I

DISCRETISATION OF THE BOUNDARY CONDITIONS OF MODEL II IN THE EXAMPLE TREATED

Upstream boundary condition. The upstream boundary condition is given by :

$$G(0,t) = G_1(t) \qquad (54)$$

where $G_1(t)$ is the known solid discharge given at point $j = 1$ of the model. Eq. 54 may by presented in the form of Eq. 31, i.e. :

$$\Delta z_1 = E_1 \Delta y_1 + F_1 \qquad (55)$$

The solid transport formula is a function of type :

$$G_1^n = G_1(Q, y, z, t_n) \qquad (56)$$

known at time $t_n = n.\Delta t$. At the next moment $t_{n+1} = t_n + \Delta t = (n+1)\Delta t$ it will be found that :

$$G_1^{n+1} = G_1^n + \Delta G_1 = G_1^n + \frac{\partial G_1}{\partial y}\Delta y_1 + \frac{\partial G_1}{\partial z}\Delta z_1 \qquad (57)$$

whence :

$$\Delta z_1 = -\frac{\dfrac{\partial G_1}{\partial y}}{\dfrac{\partial G_1}{\partial z}}\Delta y_1 + \frac{G_1^{n+1} - G_1^n}{\dfrac{\partial G_1}{\partial z}} \qquad (58)$$

and Eq. 55 is again found with :

$$E_1 = -\frac{\dfrac{\partial G_1}{\partial y}}{\dfrac{\partial G_1}{\partial z}} \quad ; \quad F_1 = \frac{G_1^{n+1} - G_1^{n}}{\dfrac{\partial G_1}{\partial z}} \tag{59}$$

Relationships 59 whose terms are known since $G_1(t)$ is given, permit beginning of the recurrence process described in the chapter SOLUTION OF THE ALGEBRAIC SYSTEM.

Conditions at downstream boundary. The downstream boundary is stated as :

$$z(L,t) = z_N = \text{const} \tag{60}$$

where z_N is the bed level at the last point of the model. This condition states the fact that deposition is impossible at this point and thus :

$$\Delta z_N = 0 \tag{61}$$

Using Eq. 31 :

$$\Delta z_N = E_N \, \Delta y_N + F_N \tag{62}$$

is found :

$$\Delta y_N = -\frac{F_N}{E_N} \tag{63}$$

Eq. 63 may be used to find $y_N^{n+1} = y_N^{n} + \Delta y_N$ and, subsequently, to start the recurrence process for finding levels y_j^{n+1}, z_j^{n+1} at points $j = N-1, N-2, \ldots\ldots, 1$.

APPENDIX II

ANALYSIS OF THE LINEAR SYSTEM 47 WITHOUT FRICTION

Numerical stability. Given $P_4=0$ in Eqs. 52, 53 it may be stated that :

$$\exp(-\mathrm{Im}\beta\Delta t)\cos(\mathrm{Re}\beta\Delta t) = 1 - \frac{P_1^2 r^2 \theta \tan^2\alpha}{P_1^2 r^2 \theta^2 \tan^2\alpha + P_2^2} \tag{64}$$

$$\exp(-\mathrm{Im}\beta\Delta t)\sin(\mathrm{Re}\beta\Delta t) = \frac{P_1 P_2 r \tan\alpha}{P_1^2 r^2 \theta^2 \tan^2\alpha + P_2^2} = \varphi \tag{65}$$

whence :

$$\tan(\mathrm{Re}\beta\Delta t) = \frac{P_1 P_2 r \tan\alpha}{P_1^2 r^2 \theta(\theta-1)\tan^2\alpha + P_2^2} = \psi \tag{66}$$

Let $(P_1/P_2).\, r \tan\alpha = T$

$$\text{Then } \varphi = \frac{T}{T^2\theta^2 + 1} \quad ; \quad \psi = \frac{T}{T^2\theta(\theta-1) + 1} \; ; \tag{67}$$

$$\sin(\mathrm{Re}\beta\Delta t) = \frac{\psi}{\sqrt{1+\psi^2}} \tag{68}$$

Substituting Eq. 68 in Eq. 65 gives :

$$\exp(-\mathrm{Im}\beta\Delta t) = \frac{\varphi.\sqrt{1+\psi^2}}{\psi} \tag{69}$$

The necessary condition for numerical stability is written from Eq. 69 :

$$\frac{T}{T^2\theta^2+1} \cdot \frac{T^2\theta(\theta-1)+1}{T} \cdot \left\{ 1 + \frac{T^2}{[T^2\theta(\theta-1)+1]^2} \right\}^{1/2} \leqslant 1 \qquad (70)$$

whence :

$$0.5 \leqslant \theta \leqslant 1.0 \qquad (71)$$

. **R_1 convergence coefficient.** For the frictionless case, the analytical solution (Eq. 46) gives $\mathrm{Im}\beta = 0$, thus $R_1 = \varphi(1+\psi^2)^{1/2}\,\psi^{-1} \neq 1$. While for $\theta = 0.5$ is found : $\varphi = T/(1.0 + 0.25\,T^2)$, $\psi = T/(1 - 0.25\,T^2)$ and :

$$R_1 = \frac{1 - 0.25\,T^2}{1 + 0.25\,T^2} \sqrt{\frac{(1 - 0.25\,T^2)^2 + T^2}{(1 - 0.25\,T^2)^2}} = 1 \qquad (72)$$

. **R_2 convergence coefficient.** From Eq. 66 and Eq. 46 it will be found that :

$$R_2 = \frac{\arctan(\mathrm{Re}\beta\Delta t)\quad(\sigma\Delta t)^{-1}}{P_1\,.\,P_2^{-1}} \neq 1 \qquad (73)$$

But if $\theta = 0.5$ then :

$$\psi = \tan(\mathrm{Re}\beta\Delta t) = \frac{T}{1-(0.5T)^2} = \frac{2\,.\,(0.5T)}{1-(0.5T)^2} \qquad (74)$$

If $0.5\,T = \tan\mu$, then $\psi = \tan(2\mu)$ and $\arctan(\mathrm{Re}\beta\Delta t) = 2\mu = 2\arctan(0.5\,T)$, whence :

$$R_2 = \frac{2\arctan\left[\frac{P_1}{2P_2}\,r\,\tan\left(\frac{\sigma\Delta x}{2}\right)\right]}{\frac{P_1}{P_2}\,.\,\sigma\Delta t} \neq 1 \qquad (75)$$

The sole case of coincidence between the « numerical » and « analytical » celerities occurs when $P_1 = P_2$ and $r = 2\Delta t/\Delta x = 2$ since in this case :

$$R_2 = \frac{2\arctan\left[\tan\left(\frac{\sigma\Delta x}{2}\right)\right]}{\sigma\Delta t} = \frac{2\sigma\Delta x}{2\sigma\Delta t} = 1 \qquad (76)$$

APPENDIX III

REFERENCES

1. de Vries, M., « Considerations about non-steady bed-load transport in open channels », **Eleventh Congress of the International Association for Hydraulic Research**, Leningrad, USSR, 1965.

2. Vreugdenhil. C.B. and de Vriès, M., « Computations on non-steady bed-load transport by a pseudo-viscosity method », **Delft Hydraulics Laboratory Publ. N° 45**, Delft, Sept. 1967.

3. Daubert, A. and Lebreton, J.C. « Etude expérimentale et sur modèle mathématique de quelques aspects de processus d'érosion des lits alluvionnaires en régime permanent et non-permanent », **Twelfth Congress of the International Association for Hydraulic Researd**, Fort Collins, U.S.A., 1967.

4. Kyozo Suga, « On the simulation of river-bed variation by characteristics », **Thirteenth Congress of the International Association for Hydraulic Research**, Kyoto, Japan, 1969.

5. Perdreau, N. and Cunge, J.A. « Sédimendation dans les estuaires et les embouchures - bouchon marin et bouchon fluvial », **Fourteenth Congress of the International Association for Hydraulic Research**, vol. 4, paper D6, Paris, France, 1971.

6. Meyer-Peter, E. and Müller, R., « Formulas for Bed-Load Transport », **Second Congress of the International Association for Hydraulic Research**, Stockholm, Sweden, 1948.

7. Preissmann, A. « Difficultés rencontrées dans le calcul des ondes de translation à front raide », **Eleventh Congress of the International Association for Hydraulic Research**, Seminar, Leningrad, USSR, 1965

8. Leendertse, J.J., « Aspects of a computational model for long period water-wave propagation », **RAND Memorandum RM 5294 - PR**, 1967.

9. Richtmyer, R.D. and Morton, K.W., « Solution of the Implicit Equations », **Difference Methods for Initial-Value Problems**, Second Edition, Interscience Publishers, New York, 1967, pp. 198-201.

10. Blanchet C., « Rapport général du sujet C Thème 3 », **Fourteenth Congress of the International Association for Hydraulic Research**, Paris, France, 1971.

APPENDIX IV

NOTATION

A_1, A_2 = coefficients known at the time t_n

b = width of the free surface

$\tilde{b}$ = width affected by the bed-load transport

B_1, B_2 = coefficients known at the time t_n

c = wave celerity

c_1, c_2, c_3 = characteristic directions of partial differential equations

C_1, C_2 = coefficients known at the time t_n

D = conveyance

D_1, D_2 = coefficients known at the time t_n

E_j = coefficient known for the point j at the time t_n

F = Froude number

F_j = coefficient known for the point j at the time t_n

G = solid discharge

g = acceleration due to gravity

H_1, H_2 = coefficients known at the time t_n

h = flow depth

Im = imaginary part of a complex number

i = $(-1)^{1/2}$

J = energy gradient

j = subscript indicating the computation point

k = wave number

K = overall resistance coefficient = D^{-2}

L = length of the river reach

N = number of points in the model

n = subscript indicating the time level during computation

P_1 to P_4 = coefficients of the linearised system of partial differential equations

Q = water discharge

R_1, R_2 = convergence ratios

Re = real part of a complex number

S = cross section area

t = time (independent variable)

u = mean water velocity in the cross section

x = distance (independent variable)

y = water level = $z + h$

z = river-bed level

o = subscript indicating normal flow

α = auxiliary symbol = $0.5\ \sigma\ \Delta x$

β = coefficient linked to wave frequency = $2\pi\nu$

Δ = increment of any variable between the time levels t_n and $t_{n+1} = t_n + \Delta t$

ζ = small perturbation of the river-bed level z_o in normal flow

η = small perturbation of the water level y_o in normal flow

θ = weighting coefficient in finite difference scheme

ν = wave frequency

σ = coefficient linked to wave number = $2\pi k$

oOo

TABLE 1

VARIATIONS OF SOLID DISCHARGE UPSTREAM OF THE MODEL AS A FUNCTION OF TIME

TIME in hours (1)	SOLID DISCHARGE G in cubic meters per second (2)
0	0.718
8	0.718
16	0.990
∞	0.990

TABLE 2

COMPUTATION EFFECTED WITH MODEL I AND MODEL II

RUN NUMBER (1)	MODEL USED (2)	Δt TIME STEP USED in hours (3)	WEIGHTING COEFFICIENT θ USED (4)	COMMENTS
1	MODEL I	0.25		Standard calculation
2	MODEL II	4.00	1.0	stable
3	MODEL II	100.00	1.0	stable
4	MODEL II	4.00	0.5	stable
5	MODEL II	4.00	0.4	unstable

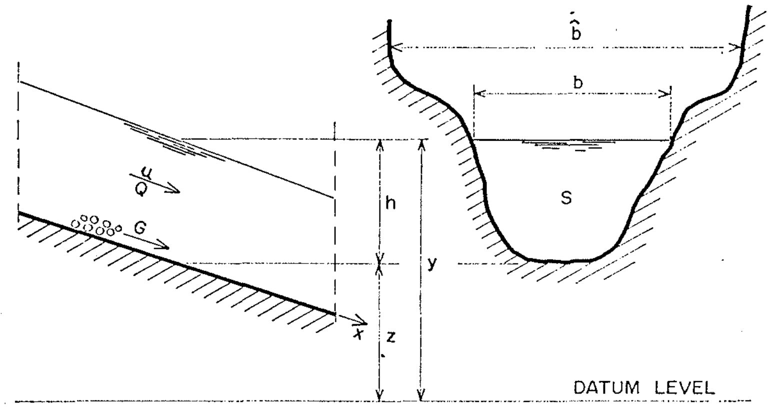


FIG. I. _ ONE - DIMENSIONAL FLOW IN A MOBILE - BED RIVER

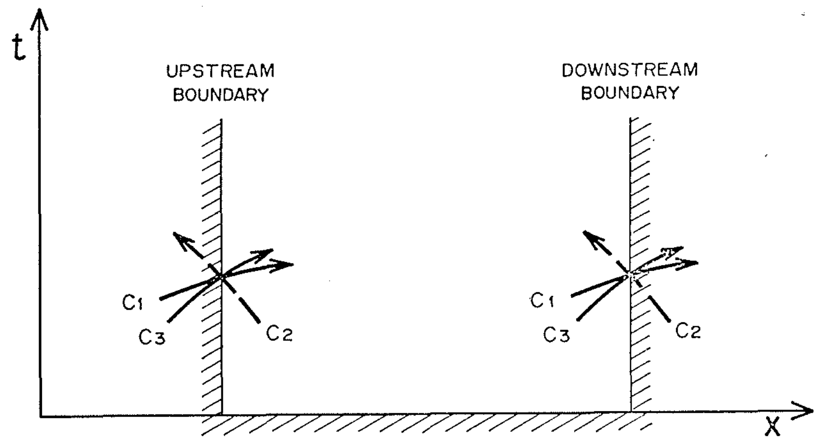


FIG. 2. _ CHARACTERISTICS AT THE BOUNDARIES FOR A VARIABLE RATE OF FLOW $Q = Q(x,t)$ *(Equ. 12)*

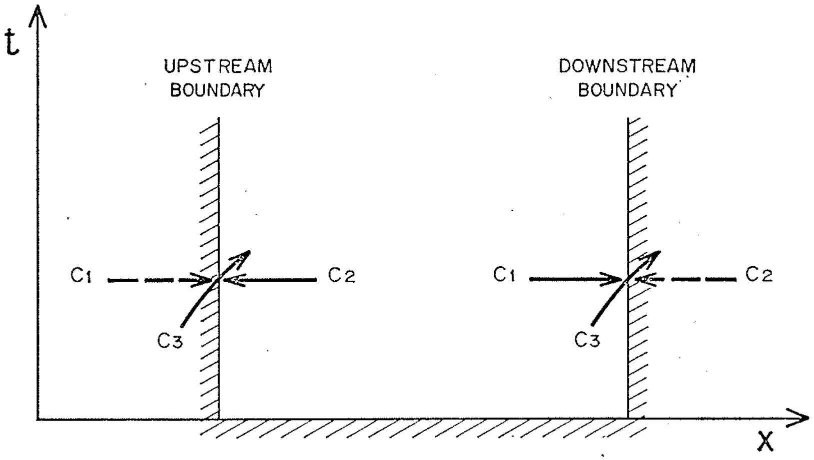


FIG.3._ CHARACTERISTICS AT THE BOUNDARIES FOR A RATE OF FLOW CONSTANT WITH TIME

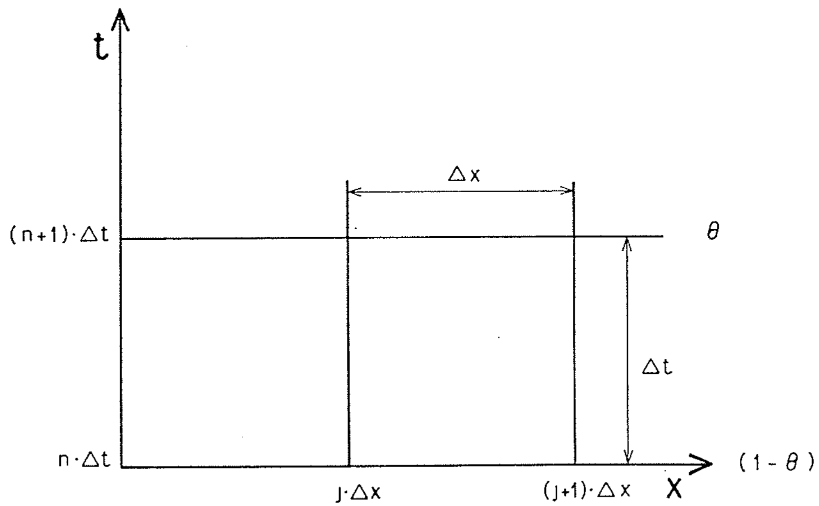


FIG.4._ IMPLICIT SCHEME OF FINITE DIFFERENCES

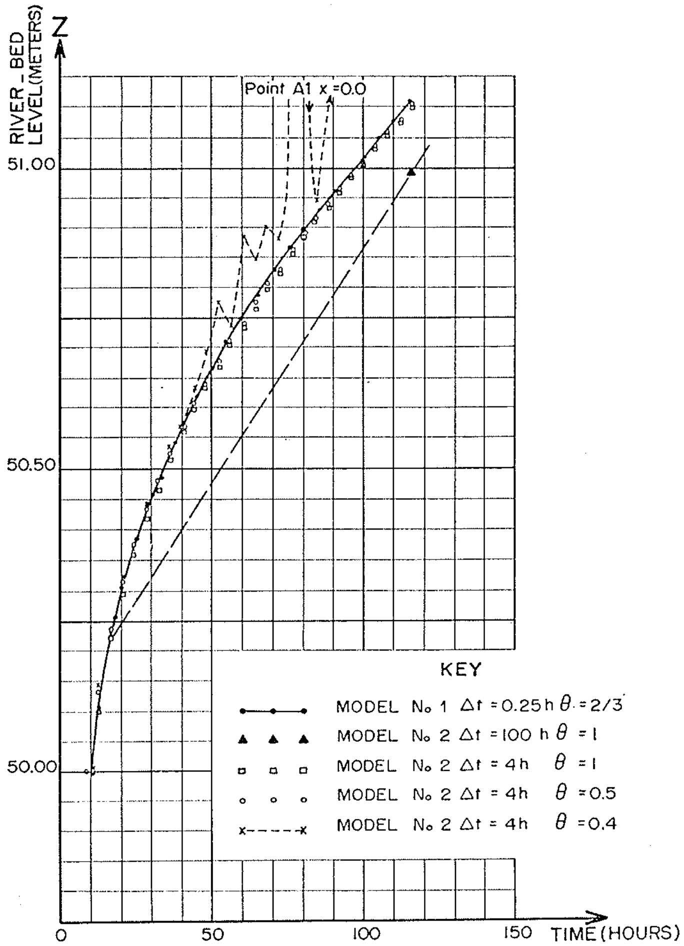


FIG.5._ RIVER_BED VARIATIONS AT THE UPSTREAM BOUNDARY

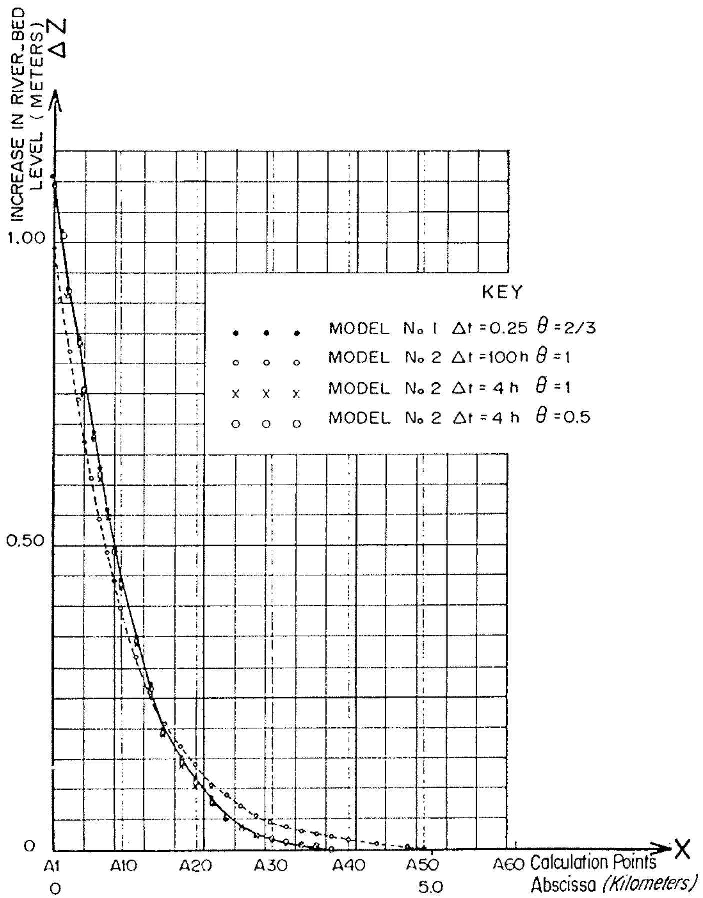


FIG.6._ RIVER_BED LONGITUDINAL PROFILE AT 116 HOURS

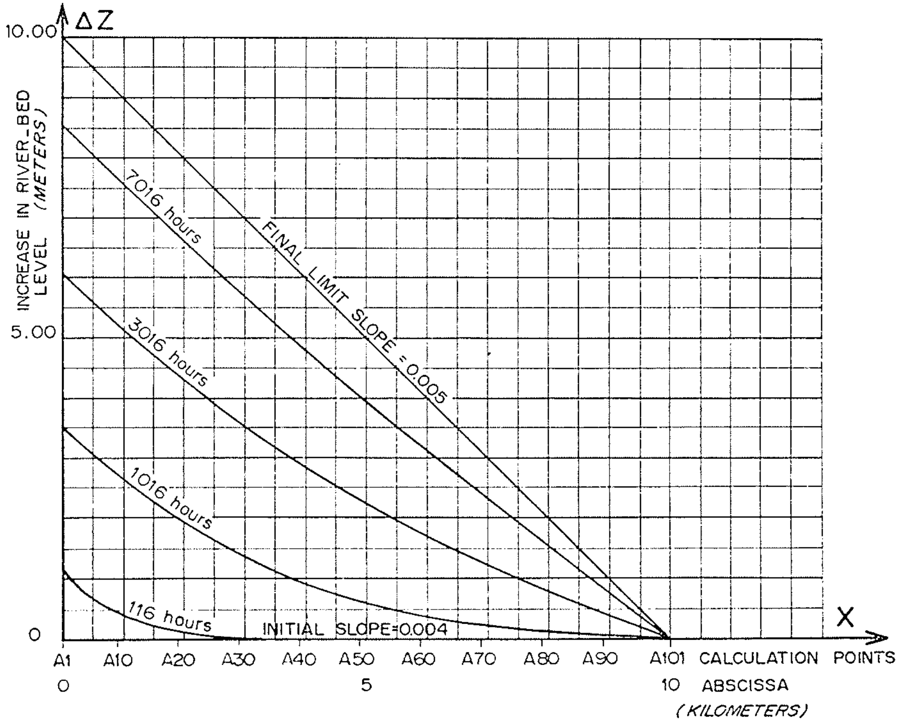


FIG.7._ LONGITUDINAL PROFILE OF THE RIVER_BED

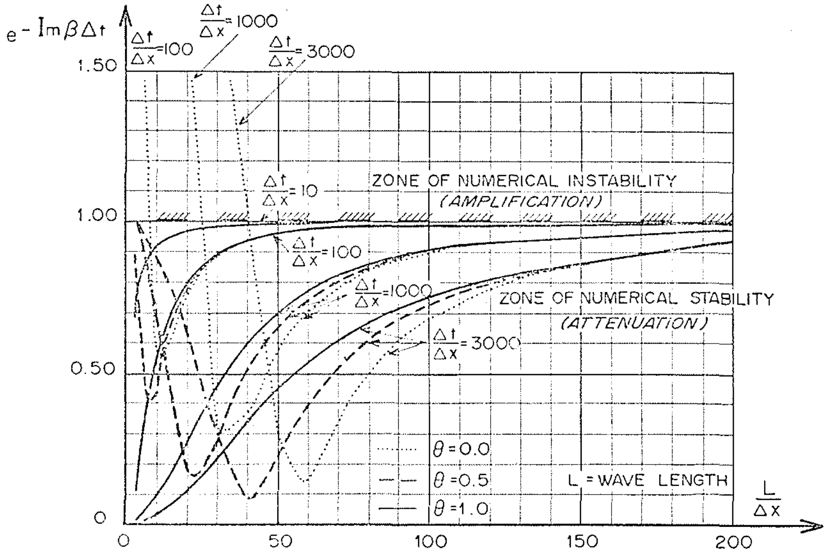


FIG.8._AMPLIFYING FACTOR *(FINITE DIFFERENCES)*

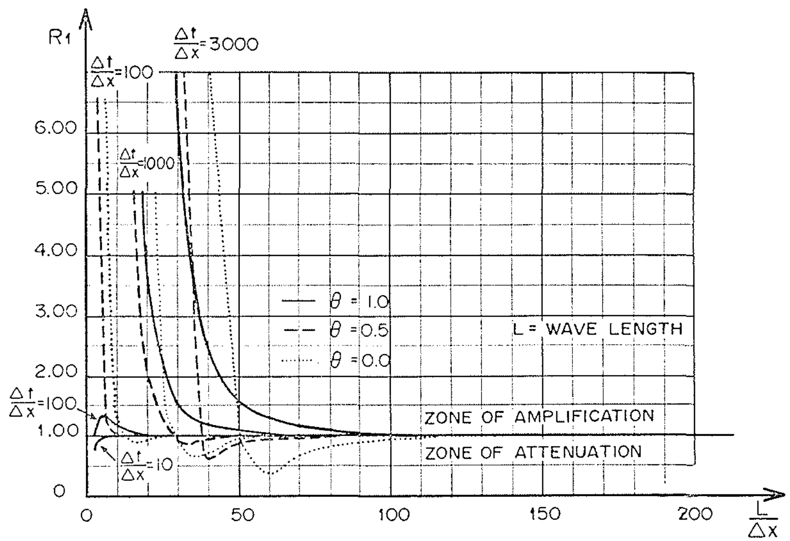
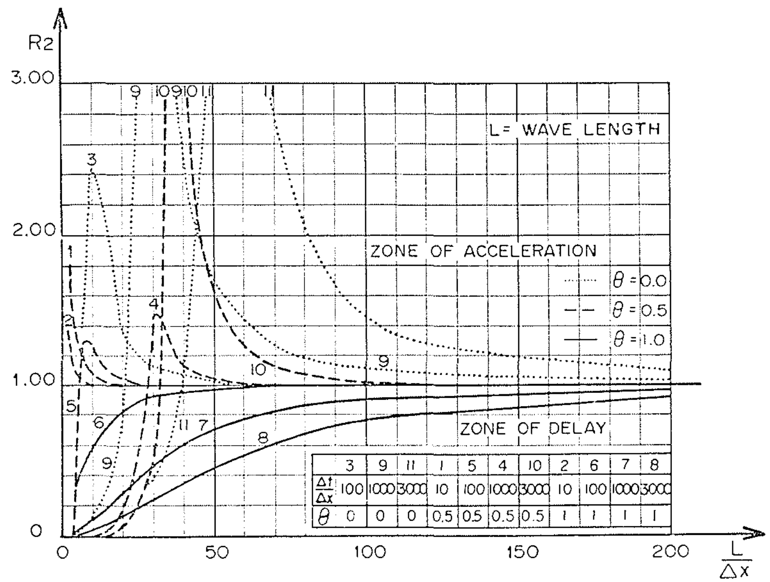


FIG.9. _ $R1 = \dfrac{\text{NUMERICAL ATTENUATION FACTOR}}{\text{ANALYTICAL ATTENUATION FACTOR}} = \dfrac{e^{-\operatorname{Im}\beta\Delta t}}{e^{-\beta\Delta t}}$



	3	9	11	1	5	4	10	2	6	7	8
$\frac{\Delta t}{\Delta x}$	100	1000	3000	10	100	1000	3000	10	100	1000	3000
θ	0	0	0	0.5	0.5	0.5	0.5	1	1	1	1

FIG.10._ $R2 = \dfrac{\text{WAVE CELERITY (NUMERICAL COMPUTATION)}}{\text{WAVE CELERITY (ANALYTICAL COMPUTATION)}} = \dfrac{\frac{\operatorname{Re}(\sigma\Delta t)}{\sigma\Delta t}}{\frac{\operatorname{Re}\beta}{\sigma}}$