Ventures into biomechanical similitude

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Like parables and fables, similarities and analogies have considerable pedagogical worth, in particular if both the analog and its counterpart possess intrinsic technical significance. For example, a lecture given at Iowa by the late Professor Antoine Craya over a quarter of a century ago is still fresh in my memory because of the manifold effect that it must have had on all who heard it. The subject was presumed to be Water Hammer, but I doubt if water was even mentioned. Instead, Craya described the circulatory system of the human body, the pumping action of the heart, and the propagation of the pulse (i.e., "blood hammer") through the elastic action of the artery walls. It did not matter that the elasticity of the blood played no role, because the analogy was otherwise sufficiently close for the comparison to be very effective. In fact, when a question was raised by a representative of our College of Medicine as to why, with advanced hardening of the arteries, bifurcations tended to be sources of physiological trouble, the answer was immediately evident from similarity of the pressure build-up at points of partial wave reflection such as pipe wyes. Through Professor Craya's lecture the students may have learned less about water hammer than if that had actually been the theme, but they acquired instead a bit of insight into the functioning of their own bodies, and—of even greater importance—they learned something about the generality of fluid mechanics.

Early in 1973 I had occasion to lecture before three successive groups of second-year civil-engineering students at the University of New South Wales on the assigned topic of hydraulic similitude. This was one of several subjects on which I was prepared to speak—but not to students just beginning hydraulics, and not three times in a row, for in such circumstances the outcome can be deadly. By chance I recalled that a former colleague, Professor Chesley Posey, had once discussed in a seminar the extent to which Jonathan Swift had understood the principle of similitude when he wrote "Gulliver's Travels". I could not remember the details of his talk, except for possible mention of a fire-fighting episode in Lilliput and a hypothetical encounter with a giant's spittle in Brobdingnag. But I did seek to emulate Professor Craya by introducing the theory of hydraulic models in terms of Gulliver, the Lilliputians, and the Brobdingnagians, and the laws that should have governed their similarity.

Now I, like Swift, could make any assumptions that I wished, though as a matter of pride I sought to be as much more scientific in my analysis as the intervening 250 years should have permitted. Unfortunately, biomechanics (or at least my knowledge of it) is by no means as old as the principle of similitude (first used by Swift's contemporary, Galileo, to estimate the strength of beams and bones), and some of the material that I presented was almost as new to me as to the students. I began on the relatively sound ground of geometric similarity, assuming with Swift that the Lilliputian male adults were on the average one-twelfth the size of Gulliver, but built in exact proportion, and the Brobdingnagians twelve times his size and also in exact proportion. This led to mention of the first towing-tank tests conducted at Iowa—not in the Hydraulics Laboratory by its research staff, but in the swimming pool at the Field House by the Department of Physical Education. Swimmers had been towed while stroking normally, or with arms only, or with legs only, the towing mechanism sometimes exerting a positive and sometimes a negative force. The data had been refined and plotted as one would those of a self-propelled model ship, in the hope of acquiring general knowledge for use in training athletes for speed swimming. But it proved to be impossible to gener-

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alize the results, for no two swimmers were sufficiently similar in their body shape to produce even comparable "bow" waves.

With this hint as to the engineering nature (i.e., open-endedness) of the similitude problem, I then raised with the class the question of the probable variation, with body size, of three basic physiological frequencies: that of the mean gait, that of the heart beat or pulse, and that of the vocal chords in normal speech. As is known from everyday experience with the animal world, geometrically dissimilar as animals may be, each of these frequencies tends to decrease with increasing size—say from mice to dogs to horses. After a little thought, the students concluded that the gait would probably be related to gravity, the pulse to viscosity (Craya to the contrary), and the vocal tone to elasticity, though in what manner and whether in the same manner for all was by no means at once apparent. Dimensional reasoning, however, permitted certain tentative conclusions to be reached.

The students appeared to be at least momentarily intrigued by the ideas, if not either lastingly convinced or potentially able to apply the notions to theorems of hydraulics still to be learned. I, on the other hand, became more and more concerned with the points that I had had to pass over lightly for lack of certainty in my own mind, and I eventually decided that the points should be further clarified—not only as a matter of principle or for the additional insight that this might give into hydraulic modeling, but in order that the field of biomechanics, with its space-travel implications, might properly be correlated with similitude theory.

As an initial step in this direction, let us see how far the basic notions of mechanical similitude can actually be carried, to the end of being able to predict from the observed mechanical behavior of a Gulliver-like being the behavior of a similar being of much larger or smaller size. If we assume complete similarity of such beings at all scales, and if we further limit the material properties under consideration to mass, weight, viscosity, and elasticity, the variables presumably necessary to describe the effects of scale on the biomechanical functions of the beings can be written as

$$P = F(L, L_n, f, f_n, \rho, \rho_n, \gamma, \mu, \mu_n, e, e_n)$$

Herein the dependent variable $P$ represents the power or rate of energy expenditure required to perform a particular rhythmic function. The linear terms should actually include all of those necessary to describe the geometry of the system; since complete similarity was prescribed, the length $L$ indicates the being's size, and $L_n$ is simply a reminder that his surroundings as well as his body members must be to scale. The term $f$ represents some frequency of biological function, and $f_n$ such other frequencies as must be taken into account. (Instead of $f$, to be sure, the velocity $V = Lf$ of bodily translation, blood flow, etc., could readily be used.) The mass density $\rho$, the weight density $\gamma$, the viscosity $\mu$, and the elastic modulus $e$ appear in multiple to indicate that more than one substance will surely be involved (however, an additional weight density would be superfluous, because $\gamma/\rho = \gamma_n/\rho_n$, and $\gamma$ is necessarily constant for any one situation.

As is well known to those in the field of mechanics [1], such a series of dimensional variables can be combined by the principle of dimensional analysis into a significant series of numerical ratios. Not only are these ratios non-dimensional, but they are generally fewer by the number of dimensional categories which the original variables involve. For the case now being considered, there are three categories: length, time, and either force or mass. In effect, three of the variables are then selected so as to include all of the categories in different combinations, and these are combined in turn with each of the remaining variables. If $L$, $f$, and $\rho$ are chosen as those that are to appear to some power (including zero) in each ratio, a routine algebraic procedure will yield the following results:

$$\frac{P}{L^3 f^3 \rho} = \Phi \left( \frac{L_n}{L}, \frac{f_n}{f}, \frac{\rho_n}{\rho}, \frac{\gamma_n/\rho_n}{\gamma/\rho}, \frac{\mu_n/\rho_n}{\mu/\rho}, \frac{e_n/\rho_n}{e/\rho} \right)$$

Cross-multiplication of the first viscosity and elasticity ratios by the reciprocals of their following counterparts will permit the function to be rewritten in the form

$$\frac{P}{L^3 f^3 \rho} = \Phi \left( \frac{L f^3}{\gamma \rho^3}, \frac{L^2 f^2}{\mu \rho^2}, \frac{L^2 f}{\mu \rho}, \frac{L f}{\mu \rho}, \frac{e}{e_n} \right)$$

According to the principle of similitude, any two or more systems described by this function will be dynamically similar if each of the nondimensional ratios that it embodies has the same numerical magnitude, respectively, in every system.

The dependent term at the left is a form of what I have named the Euler number [2]. It is evidently controlled by the independent terms at the right. The first, second, and third of these are forms of what are generally called the Froude, Reynolds, and Mach or Cauchy numbers, and it is usual to accept them as constants for conditions of similarity. Constancy of the density, viscosity, and elasticity ratios simply means that the different materials in any system (whether solids, liquids, or gases) must have proportionate properties in the respective systems, however much their absolute magnitudes may vary from system to system. Constancy of the length ratio merely specifies geometric similarity—i.e., proportionality of homologous lengths. Strictly speaking, this involves all lengths, including the diameter of the planet upon which the beings of the given size exist! (To be sure, the latter requirement actually need not be specified, as it is effectively embodied in the constancy of the Froude number.) Constancy of the frequency ratio entails kinematic similarity—i.e., proportionality of the several frequencies as the linear scale changes; therefore, the laws of variation for gait, pulse, and so on must be identical. In brief, complete dynamic similarity would require agreement of such miscellaneous details as the relative distance the being in question can jump, the number of times he breathes in one of his days, the height from which he can fall without skeletal damage, and even the comparative length of his life span.

Surely Swift would not have respected such stringent similitude requirements even had the understood them, for it was essential to his purpose that Gulliver visit the two kingdoms—not to mention several others—without actually leaving his own world. (Far from creating a pointless children's study, it should be recalled, Swift wrote for adults, to illuminate the pettinesses, exaggerations, and other foibles common to the England of his day). However, as is only too well known in mechanical modeling, if one fails to meet a single requirement, complete
dynamic similarity cannot be obtained—and approximating it artificially in one respect may cause still further departure from it in another. Thus, instead of involving the interdependent variation in linear scale, frequency of action, and material properties so vital to the attainment of complete similarity, Gulliver's travels actually occurred (a) under constant gravitational conditions, (b) in contact with the same water, air, and earth, and (c) among beings composed (presumably) of the identically constituted bones, flesh, and blood. What, then, of their various biological frequencies, depending as they all must on the same material properties in their complex relation to the linear scale?

For the conditions unwittingly imposed by Swift, the foregoing equation can be written simply as

$$\frac{P}{L^2 f^3 \rho} \neq \psi \left( \frac{L_n f_n}{L f} \right)^n, \gamma = \frac{\rho \rho}{\rho \rho}, \mu = \frac{C}{\rho \rho}, e = \frac{C}{\rho \rho}$$ \tag{4}

Evidently, the frequency of the various body functions would have to vary with $1/L^{1/2}$ to satisfy the Froude criterion of similarity, with $1/L^2$ to satisfy the Reynolds criterion, and with $1/L$ to satisfy the Mach or Cauchy criterion. Not only is this a physically impossible situation—hence the inequality sign—but it is at odds with the requirement of kinematic similarity. Evidently, true dynamic similarity between Gulliver and either the Lilliputians or the Brobdingnagians was out of the question. Whether kinematic similarity was still possible remains to be seen.

Even geometric similarity between living bodies of such different sizes is conceivable only in a creative mind like Jonathan Swift's. On the other hand, biologists have long compared the actions of large and small animals, or birds, or fish. As early as 1927, for example, Lamb and Teissier[3] assembled data on the period, or inverse frequency, of the pulse of mammals at rest for well over a thousandfold range of body mass, as plotted in Figure 1[4]. If the usual practice of assuming the mass to vary with the cube of the scale is followed, the linear relation between the period and the cube root of the mass will indicate that the scale and the frequency are inversely proportional. Upon introduction of the proportionality $f \sim 1/L$ into the Euler number $P(L^2 f^3 \rho)$, moreover, it will be found that $P \sim L^2$.

Now the power requirements of mammals, insects, birds, and fish have also been studied in detail by biologists, with the composite results shown in Figure 2[5]. Therein the energy consumed per unit body mass and distance traveled is plotted as a function of body mass for all three forms of locomotion. The propulsive efficiency (the vertical position of each line) evidently varies considerably from running to flying to swimming, but the form of the function is very nearly the same. Indeed, as a first approximation all three lines can be considered to have the $-1/3$ slope represented by the broken line. If it is again assumed that $f \sim 1/L$, it will now be seen that $P \sim E/T \sim Ef \sim L^2$ for insects, birds, and fish as well as mammals. Though no data are at hand for vocal-chord vibration, it has already been noted that the Mach criterion for similarity is in accord with the frequency ratio $f \sim 1/L$. Presumably, then, constancy of the power and frequency terms for comparable body types and materials is the extent of Swift's similitude conditions:

$$\phi \left( \frac{P}{L^2 f^3 \rho} \frac{f_n}{f} \right) = 0$$ \tag{5}

One is now inclined to wonder whether full similarity is actually important in biomechanics. The principle of similarity is used, of course, as a means of predicting from laboratory tests the behavior of a body at a different scale when the functional relationship for such behavior is not known. Sometimes only partial similarity can be offset by a partial approximation of the relationship. The inequality of Equation (4), for instance, is in some ways reminiscent of the problem that arises in the prediction of ship resistance from towing-tank tests on scale models. Both wave (gravitational) and friction (viscous) effects are involved, and hence true similarity (constancy of the Froude and Reynolds numbers) cannot be realized without varying the ratio of viscosity to density in proportion to the $3/2$ power of the scale—which is practically out of the question. Instead, since wave resistance changes rapidly with the Froude number, the model ship is towed at the speed indicated by the gravitational criterion, and the more gradual variation of boundary-layer resistance with the Reynolds number is introduced as an empirical correction when converting the model indication to the prototype prediction. The situation is indicated schematically in Figure 3 for any three-variable relationship. If true similarity is to exist between two systems, the independent variables $X$ and $Z$ must be respectively the same in both systems (represented, say, by point A), and the dependent variable $Y$ will then be fixed in magnitude. If only one variable $Z$ (for example, the Froude number) can be made the same, the true magnitude of $Y$ (the Euler number) for the prototype system must be found by use of an empirical formula between $Y$ and $X$ (the Reynolds number) for $Z = \text{constant}$. Evidently, no similarity need exist if the complete relationship among the several variables is known either empirically or analytically.

![Graph 1: Dependence of the period of the resting heart on the cube root of the body mass [3, 4]]

![Graph 2: Energy costs for swimming, flying, and running as functions of the body mass [5]]
Although the various functional relationships for the animal body are far from completely understood, certain basic parts thereof can at least be discussed in mechanical terms. Of the three forms of locomotion, swimming at some depth below the water surface should be wholly a resistance phenomenon independent of gravity. Flying, on the other hand, entails work to counterbalance both resistance and weight. In a similar manner one might reason that air resistance to animals traveling on foot should be minimal, but that the effect of gravity should increase with advancement from walking, to running, to leaping. (In the latter regard is should be noted that the unusual gait of the astronauts on the moon may well illustrate the effect of reduction in gravity with constancy of the Froude number, but that there is no apparent reason for the frequency of the pulse or vocal-chord vibration to change in proportion). That the frequency and power actually vary in a comparable manner in all three forms of locomotion with constant fluid properties would indicate that it is a question of mass acceleration of the body parts that is the determining factor rather than the kind of resistance to be overcome.

So far as internal resistance effects are concerned, the viscosity of animal blood is known to be practically constant at four times that of water regardless of body scale. The diameter of the major blood vessels may be assumed directly proportional to the scale. The kinematic factor required to form a Reynolds number must be one of the following: the frequency of the pulse, the mean velocity of the blood at some characteristic section, and the celerity of the pulse wave. The latter is of least significance in the present instance, because—witness Cray–the “blood-hammer” effect is essentially an elastic phenomenon involving the Cauchy rather than the Reynolds number. Based on the mean velocity of the blood stream itself, which—if anything—increases only slightly with the scale, the customary Reynolds number \( \rho VL/\mu \) for steady flow varies approximately with the first power of the scale \[4\]. Based on the frequency of the pulse, the Reynolds number \( \rho L^2 f/\mu \) likewise varies with the first power. The resistance parameter is, in a word, by no means independent of the scale. This is really only logical, for the likelihood of eddy formation in the heart and the subsequent onset of turbulence in the aorta—a true Reynolds-number effect—should increase with body size. The modeling of arterial flow at other than natural scale evidently requires the selection of fluid and material properties in accordance with the Reynolds and Cauchy similarity criteria.

Two further aspects of elastic behavior deserve note, in addition to remarks already made about the celerity of the pulse wave and the compatibility of the Mach or Cauchy number with the frequency relationship \( f \sim 1/L \). One is the well-known tendency of the pitch of the voice to change inversely with the density of the gas that is breathed, in accord with the Mach criterion of similarity: for example, when helium is substituted for nitrogen in deep-sea diving. The other is the significance of the Cauchy form of the number when used in connection with skeletal behavior. If the ultimate strength of bone structure \( \rho_u \) is substituted for the elastic modulus, and the product of length and frequency is replaced by a velocity, constancy of the resulting Cauchy-like number \( \rho V^2/\rho_u \) indicates why (as sensed by Galileo) it is so much more dangerous for elephants than for mice to fall from a given height!

Though the understanding of even these three primary aspects of mechanical similarity is by no means a simple matter, it must be realized that there are still further physical and biological characteristics to be considered, such as surface tension, permeability, thermal and electro-chemical properties of the anatomy and its surroundings, and probably many others that have not yet come to light\[5\]. The attainment of similarity with respect to any one of these may well be without practical significance. On the other hand, the study of limited aspects of any given problem can contribute markedly to eventual understanding of the problem as a whole. Examples of simplified studies include the flow of plasma and red blood cells in the capillaries, or the flow of tears in the eye; such laboratory models are often much larger than their prototypes, and the selection of the different fluids demanded by similarity criteria is far more feasible than in the towing tank. However, the degree of similarity attainable will remain definitely limited in the number of variables that can be simulated at the same time.

Whereas the initial goal of this paper—determination of the similarity law for frequencies of gait, pulse, and vocal chords in Gulliver’s several kingdoms—has been accomplished, it was admittedly done more empirically than analytically. Furthermore, difficult as the application of similarity principles may be in mechanics, the additional complexities of biological phenomena now appear to make complete biomechanical similarity forever out of the question. It may hence be generally necessary to seek only partial similarity, correcting for the resulting discrepancies by empirical formulas for the conditions not simulated, or simply disregarding them. The ideal, of course, would be the determination of complete functional relationships for the phenomena under consideration. Till this millennium (even more remote than that of complete similarity) is reached, conditions of imperfect similarity must continue to be tolerated.

At successive stages in its preparation, the manuscript of this article was critically examined by T.L. Shaw of the University of Iowa, and T.Y. Wu of the California Institute of Technology, whose helpful comments I greatly appreciated. Just as the final stage was reached, my attention was called to an extensive discussion of biological similarity by the mathematician W.R. Stahl \[6\]. Because of that author’s attempt to be all-encompassing in this treatment, however, my article has little in common with it beyond its emphasis on the complexity of the subject.
References