Stability, buckets and flotation

Étude de la stabilité des seaux, et des corps flottants

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Open containers of liquids, such as ordinary buckets, have stable and unstable configurations like those for floating bodies. Parallel analyses of the two systems yield comparable results, some of which are intriguing. The analyses are simple for elementary shapes, and among these, circular and square cylinders are both familiar and representative.

The stability of floating bodies is a classic problem with a well known solution which has not been extended to include the stability of a bucket with a pivoted handle. The latter appears to be recognized in practice, however, in that most paint cans and milk buckets have comparatively stable configurations. Both problems can be treated by the familiar method in which the body of the bucket is tipped slightly and the system is then examined to see whether it tends to return to equilibrium or continue to overturn.

A cylindrical bucket is shown both upright and tipped in Figure 1. If the weight of the bucket is neglected, the problem reduces to a consideration of two opposing couples. One is that caused by the small segment of water which seems to move from one side to the other when the bucket is tipped. It is an overturning couple of $2F\bar{x}$, in which F is the weight of the segment or wedge and $\bar{x}$ is the distance of its center of gravity from the central axis, or axis of tipping, in the free surface. The other is the righting couple of the weight, W, at its original position, and T the equal supporting force, which is $W\bar{y} \sin\alpha$, in which $\bar{y}$ is the distance from the point of support to the center of gravity of the whole body of liquid and α is the small angle of tipping.

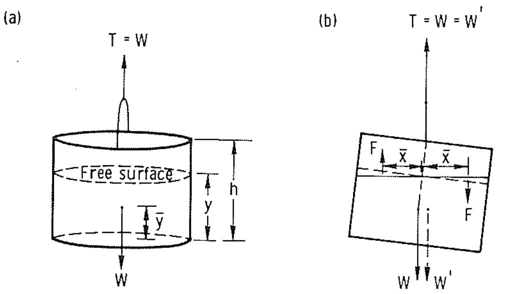

Figure 1 – Open bucket supported by conventional pivoted handle : (a) in equilibrium position, (b) slightly tipped.

An alternative statement of the underlying concept is simply that stability depends upon whether the center of gravity of the liquid in its new configuration is further to the left than the position of support (unstable) or not so far (stable). Still a third approach requires the introduction of the concept of the metacenter.

However the problem is viewed, the condition of stability depends upon the relationship,

$$\alpha \gamma \forall (h - \bar{y}) \gtreqless \gamma \int_A \alpha x^2 \, dA \qquad (1)$$

in which γ is the unit weight of the liquid, h is the vertical distance from the bottom of the container to the point of support, $\bar{y}$ is the vertical distance from the bottom to the center of gravity of the liquid volume $\forall$, dA is an element of the free surface and x is the distance from the centroidal axis in the free surface to the elementary volume, $\alpha \times dA$, within the small wedge. The upper condition in Eq. 1 is stable, and the lower one unstable with the equality representing neutral stability as usual. The integral $\int_A x^2 \, dA$ is by definition equal to I_g, the centroidal moment of inertia of the free surface. The condition equation reduces to

$$\forall (h - \bar{y}) \gtrless I_g \qquad (2)$$

The result expressed in Eq. 2 is much like the corresponding relationship for the stability of a floating body :

$$\forall(\bar{y} - y_b) \gtrless I_g \tag{3}$$

in which $\bar{y}$ and y_b represent the vertical positions of the centroids of the body and the displaced liquid, respectively. The differences in the two results, Eqs. 2 and 3, are (1) the vertical position of the point of support replaces that of the center of gravity of the floating body; (2) the volume of liquid in the container replaces the volume of liquid displaced by the floating body; and (3) the conditions of stability are reversed. A disk-shaped bucket tends to be unstable whereas a disk-shaped body floats stably, the axis of rotation being vertical in each instance.

The condition of neutral stability for circular (and weightless) buckets is

$$\frac{y}{h}\left(1 - \frac{y}{2h}\right) = \frac{1}{4}\left(\frac{r}{h}\right)^2 \tag{4}$$

an ellipse which is shown by the broken line in Figure 2, $(\bar{y} = y/2)$. The bucket shown there, for which $h = r$, is stable if $y/h > 1 - 1/\sqrt{2}$ or 0.293. The depth of water shown is stable whereas one shallower than the limiting value would cause tipping. A bucket can extend above the point of pivoted support, and the upper branch of the ellipse serves for this case. It involves, as usual, the other root of the quadratic equation for which $y/h < 1 + 1/\sqrt{2}$ or 1.707. As r/h increases, the stable region diminishes, and no depth of liquid is stable if $r/h > \sqrt{2}$. Other prismatic buckets give similar results but with other coefficients to reflect the appropriate moments of inertia. For a square bucket with a width of s,

$$\frac{y}{h}\left(1 - \frac{y}{2h}\right) = \frac{1}{12}\left(\frac{s}{h}\right)^2 \tag{5}$$

Non-prismatic buckets lead to results of different forms. Simple types are buckets which are shaped like cones, Figure 3 (a), or wedges, Figure 3 (b). Results for the latter differ depending upon whether it is supported at AA or BB. For the one shaped like an ice cream cone, the trend of neutral stability is given by

$$\frac{y}{h} = \frac{4/3}{1 + (r/h)^2} \tag{6}$$

in which y and h are the depth of water and height of support as before. The result is depicted in Figure 4. If $r/h = 1$, y/h cannot exceed 2/3 without tipping, a result in contrast to that for prismatic buckets which tend to be stable if full or nearly full.

The phenomenon is easily demonstrated with buckets which are made of thin sheet metal and are large enough that the effect of the weight of the bucket is negligible. Instability is usually avoided in practice, but does occasionally occur. Campers have reported shallow cylindrical teakettles that are stable when full but tend to put out the fire as part of the water is poured off. Also, paint cans and ordinary teakettles become unstable when nearly empty. They may not tip so far as to spill, however, as they usually find a new position of equilibrium in a tipped position. The weight of the can tends to stabilize the system and is not negligible if the depth of water is small.

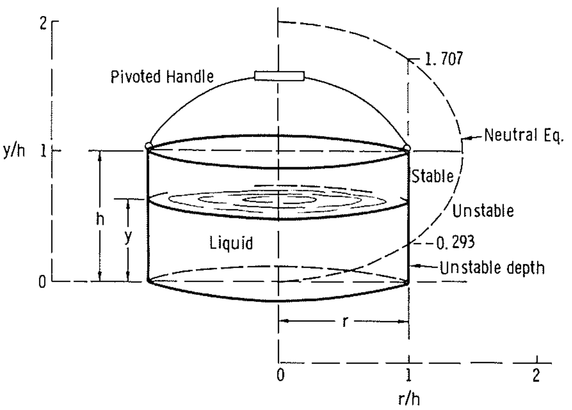

Figure 2 – Configuration of stability of a partially filled circular container; $r/h = 1$ for the special case shown.

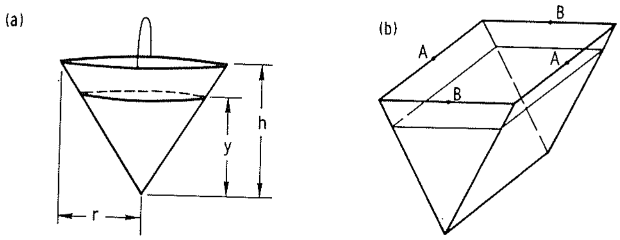

Figure 3 – Non-cylindrical buckets (a) conical and (b) prismatic.

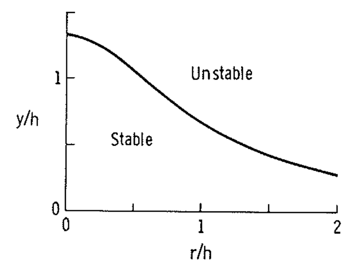

Figure 4 – Configuration of stability for a conical bucket.

Unexpected results occur for floating bodies even though the study of their stability is classical [1,2]. Lamb [2] deprecated his own treatment of some of these examples in citing a much earlier Latin quotation of Huygens (1629-95) who felt that such work lacked practical value and was therefore not worthwhile. Yet few people know that a homogeneous circular cylinder, among the simplest of floating bodies, can float stably with its axis of rotation (1) vertical, (2) horizonal, (3) neither or (4) both, depending upon its shape (D/L) and specific gravity s. Results of straight-forward applications of Eq. 3 to this case are shown in Figure 5. Experience tells us that flat disks float with their axes vertical, as mentioned, and log-like cylinders with their axes horizontal. Less well known is the existence of a range, the lens near the center of Figure 5, for which neither position is stable. If the ratio of diameter to length, D/L, changes progressively within this interval, the inclination of the axis varies continuously from one limiting value to the other. Finally if the cylinder is made of material which is either very light or nearly as heavy as the liquid, it will float stably in either position, Figures 6 and 7.

A long prism with a square cross section introduces an additional anomaly, parts of which are described in some text books [3,4]. Such a prism will float stably with one diagonal vertical for the condition

$$\frac{9}{32} < s < \frac{23}{32}$$

and with two sides vertical if

$$\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) < s < 1 \quad \text{or} \quad 0 < s < \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$$

These values also leave gaps; neither position is stable if $.211 < s < .289$ or $.729 < s < .789$. And these gaps are also bridged by progressive tipping from one position to the other as s varies within the two intervals. A square prism also tips about an axis normal to its length if it is progressively shortened. The resulting pattern of stability is depicted in Figure 8 and is like that for the circular cylinder, in part. A square block of the right length and the right specific gravity can fall in both kinds of gaps so as to produce unsymmetrical tipping about two axes as shown for the cubical child's block in Figure 9.

The stability equations lead to equations of ellipses for the curve of neutral stability for both prismatic bodies and buckets. Some of these are depicted as regular ellipses in Figure 2, 5 and 8 in which the variables are s and D/L or y/h and D/h. Other conditions introduce the elliptical equation in terms of L/D instead of D/L as for the curves which are open to the right in Figure 5 and 8; or in terms of $\sqrt{s}$ instead of s for the square prism with one diagonal vertical. The latter produces the cusp at $s = 0.5$, $d/D = 1$, and it results from the change in geometry as the horizontal diagonal is submerged or exposed.

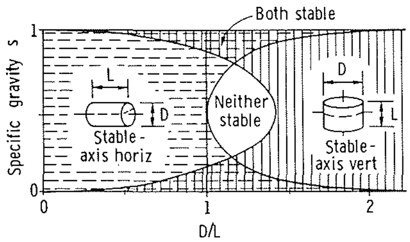

Figure 5 – Zones of stability for a floating circular cylinder.

a)

b)

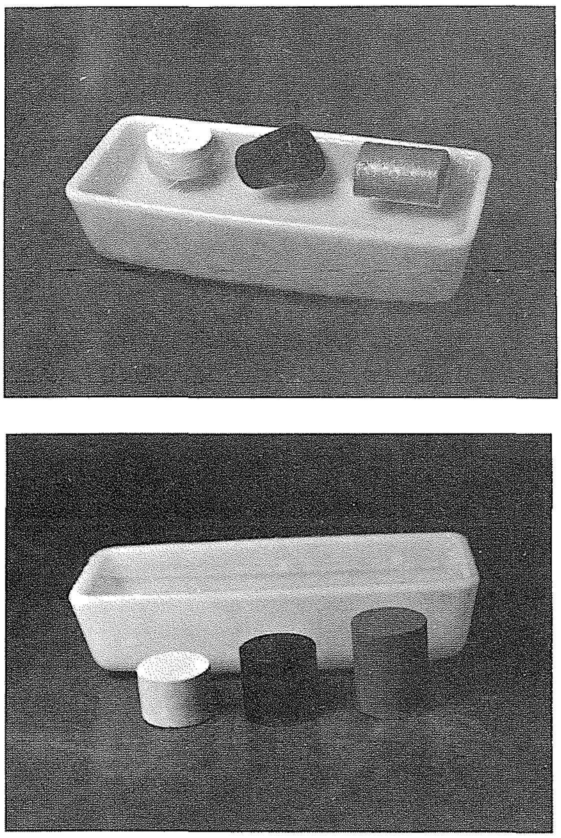

Figure 6 – Floating cylinders for which $s = 0.66$ and $D/L = 0.8$, 1.1 and 1.4, (a) in and (b) out of water.

a)

b)

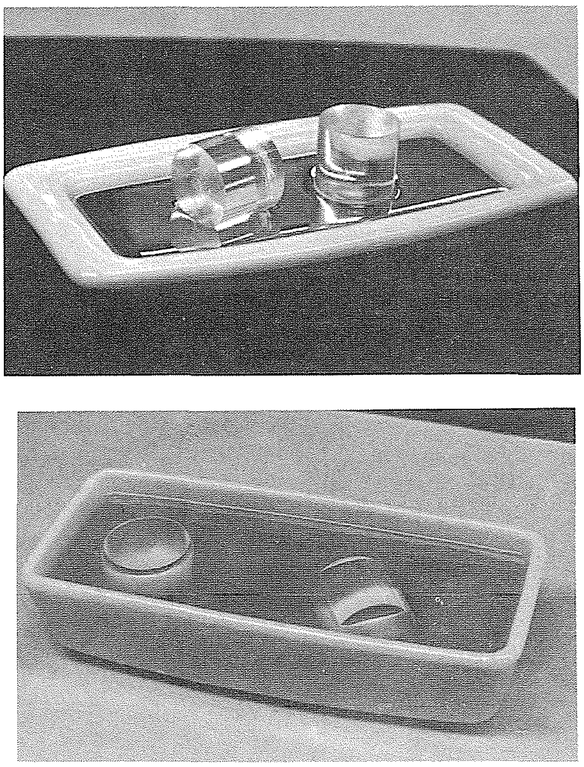

Figure 7 – Lucite cylinders, $D/L = 1.1$ floating (a) on mercury ($s \approx 0.1$) and (b) in a solution of CCl_4 and toluene ($s \approx 0.9$).

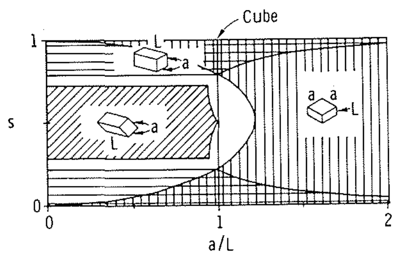

Figure 8 – Zones of stability for a floating block with a square cross section.

Figure 9 – A child's cubical block floating in water ($s \approx 0.7$).

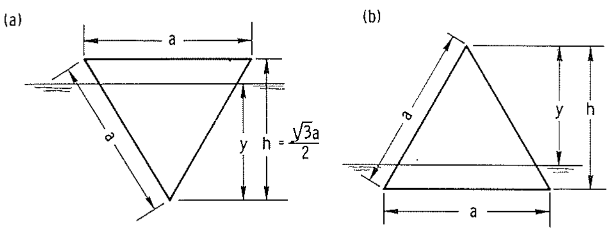

Figure 10 – Section of long triangular prisms floating stably with (a) one face and (b) one edge uppermost.

A long prism with an equilateral triangle for a cross section floats stably with a flat side uppermost as in Figure 4 (a) if $y/h > 3/4$. It also floats with an edge uppermost as in Figure 4 (b) if $y/h > 3/4$. The seeming anomaly in these answers being identical disappears since y is below the surface in Figure 10 (a) and above it in Figure 10 (b). Since $s = (y/h)^2$ in the former case, and $s = 1 - (y/h)^2$ in the latter, the two results define quite different domains of stability. A triangular prism will float stably with a flat side uppermost if $s > 9/16$ and with an edge uppermost if $s < 7/16$ [4]. The gap between the two and the progressive tipping within it occur once more, in this instance for the range $7/16 < s < 9/16$. A result comparable to those of Figure 5 and 8 could also be presented for this shape.

Gaps separating stable zones appear to be characteristic of the various occurrences. They are bridged by incorporating the more complex geometries of bodies tipped through finite angles and performing the related algebra. The analyses are, however, inherently different from those reflected by Eqs. 2 and 3. They correspond to conditions of static equilibrium rather than to zones of stability. That is, the body tips enough to align vertically the centroids of the body and the displaced liquid. The counterpart to this condition for a bucket is the nearly empty teakettle which tips through an angle such that the centroid of a segment of water falls directly under the handle, as suggested in a foregoing paragraph. All of these intermediate cases are characterized by a transition in geometry as an edge is submerged or exposed. The gaps reflected by variations of the relative length of the cylinder disappear symmetrically for floating bodies if s is either very small or nearly equal to one. In these ranges either regular position (disk-like or rod-like) is stable and all tipped positions are unstable.

Buoyancy, metacenters and stability of flotation are usually thought of as subjects that began with Archimedes and were pretty well finished by scientists working in the 19th century. Much was done by marine engineers in designing ships or by hydraulic engineers who posed and solved various problems involving some of the simple cases mentioned herein. Some of them appear to have been treated, more as exercises to tax the student than as parts of general results [5]. The results indicated in Figures 5 and 8 are not well known. Furthermore, no one appears to have considered the analogous case of the bucket. The absence of any treatment of the latter is all the more surprising since the problem it represents is a practical one which many people have experienced. Instability could occur catastrophically if efforts were made to lift a vat of molten metal for which the configuration is unstable. Perhaps this presentation, one of few papers to be written during this century on elements of hydrostatics, will bring forth references to earlier studies of container stability.

Références

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Notation

The following symbols are used in this paper :

A	= surface area
D	= diameter
F	= force
h	= height of container
I_g	= moment of inertia about gravity axis
L	= length
r	= radius
s	= ratio of unit weight of solid to unit weight of liquid
T	= supporting force
$\forall$	= volume of liquid or displaced liquid
W, W'	= weight
x	= horizontal distance
$\bar{x}$	= horizontal distance from axis to centroid
y	= vertical distance
$\bar{y}$	= vertical distance to centroid
α	= small angle of tipping
γ	= unit weight