ON FLOW IN ESTUARIES

PART I

A critical review of some studies of slightly stratified estuaries

PART II

A slightly stratified turbulent flow

by

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Nous reprenons, pour cet article, la présentation inaugurée dans le N° 1-1972, qui permet une publication rapide d'articles présentant une partie mathématique importante.

LA HOUILLE BLANCHE.

ON FLOW IN ESTUARIES
PART I: A CRITICAL REVIEW OF SOME STUDIES
OF SLIGHTLY STRATIFIED ESTUARIES

I.- INTRODUCTION.

The flow in estuaries has been studied for many years, and significant progress has been made in predicting the behaviour of sharply stratified (salt wedge) and unstratified (well mixed) types. In between these limiting cases are estuaries in which partial mixing of salt and fresh water occurs and in which there are continuous vertical and horizontal salinity gradients, typical of which is the coastal plain estuary of Pritchard.

Pritchard (1), (2) developed simplified equations to describe a coastal plain estuary, and these equations have been further developed and partially solved by Rattray and Hansen in a series of papers (3), (4) and (5). Many coastal plain estuaries and some other classes of estuaries are slightly stratified, as defined in later sections, and for these cases the physical basis of Pritchard's model is reviewed. In the latter part of the paper the simplified equations used by Hansen and Rattray are obtained in a more systematics manner and the solutions they presented are discussed.

In a later paper in this series we will consider a more rigorous derivation of the mathematical model.

2. Physical description

Cameron and Pritchard (6) define an estuary as follows: "an estuary is a semi-enclosed coastal body of water having a free connection with the open sea and within which the sea water is measurably diluted with fresh water deriving from land drainage". In this paper we discuss estuaries which are only slightly stratified, i.e. the vertical salinity gradient is small at any point of the estuary. This defination is more restrictive than that considered by Pritchard as it excludes estuaries where the vertical salinity gradients are locally large even if their average over the depth is small. This type of estuary is usually produced when the mixing of fresh and salt water results mainly from the turbulence due to tidal motion. Other factors which contribute to the mixing are the shear stress at the bottom and the wind stress at the surface.

The basic non-tidal or average circulation in estuaries of this type is shown in Fig. 1. The fresh water enters at the right and is gradually mixed with the salt water which enters at the left, mainly at the bottom. The average currents are due to the entry of the fresh water and to the density difference between fresh and salt water. Thus the analysis constitutes a problem with features analogous to both free and forced convection. The problem has been considered by several authors, e.g. Arons and Stommel (7), Ippen and Harleman (8), and Abbott (9), who have in their analyses suppressed one aspect of the problem by assuming a one-dimensional situation. The first treatment which considered both aspects of the problem was given by Hansen (4) and Hansen and Rattray (5), but they did not consider the whole length of the estuary. A much earlier analysis by Ekman (10) gave realistic flow patterns only because of wind stress, the gravitational term having been omitted in his approximate equations of motion.

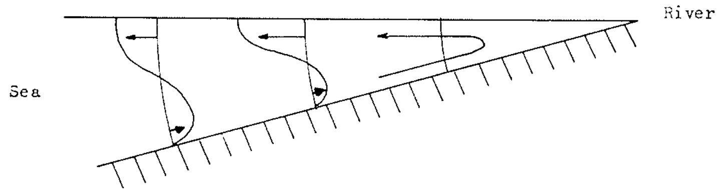


FIGURE 1 SKETCH SHOWING NON-TIDAL CIRCULATION AFTER EKMAN (10).

3. Mean equations of motion and continuity

In this section we simplify the mean equations of motion and continuity by making some assumptions about the correlations between the mean velocity, the oscillating tidal velocity, and the turbulent fluctuations.

Notation Used in Text

A	Kinematic eddy viscosity; $A^* = A/A_o$ where A_o is a reference value of A.
B	Breadth of estuary, taken as a function of x only; $B^* = B/B_o$ where B_o is a reference value of B.
D	Depth of estuary, taken as constant or as a function of x only.
g	Acceleration due to gravity.
k	Constant in equation (4.6), with units of $(\text{salinity})^{-1}$.
K	Diffusion coefficient for salt; K_z, K_x, K_{xo} being vertical, longitudinal and a reference value respectively.
L	Estuary length.
L_o, L_1, L_2	Length scales used in equation (5.8); L_o is used in Section 6 with a different definition.
p	Pressure.
R	River discharge per unit time.
S	Salinity (instantaneous value); S_T tidal component of S; S' turbulent component of S; $S^* = S/S_o$ where S_o is a reference salinity.
t	Time.

u	Velocity (instantaneous value); u' turbulent component of u; $\underline{u}$ velocity vector.
U	Tidal component of u, having an amplitude U_o.
x	Longitudinal co-ordinate, measured positive seawards; x_o is any reference value of x.
y	Lateral co-ordinate.
z	Vertical co-ordinate, positive downwards from the surface.
α	Exponent.
ε	$\varepsilon = k\, S_o$
ζ	Non-dimensional longitudinal coordinate.
η, η_1, η_2	Non-dimensional depth co-ordinates.
θ	A component of salinity.
ν	Constant.
ξ	Non-dimensional longitudinal co-ordinate.
ρ	Density; ρ_f density of freshwater.
σ	Variable proportional to salinity difference.
τ	Wind stress.
ψ	Non-dimensional stream function.
Ψ	Stream function; $\Psi^* = \Psi/\Psi_o$ where Ψ_o is a reference value of Ψ
Ω	Angular velocity of the earth.
$\langle\rangle$	or superscript $^{-}$ Time average over one or more tide cycles.

Consider a long and fairly narrow estuary, terms whose meaning will be defined later. Take the origin of co-ordinates in the free surface at the upstream limit of the estuary and let x be the longitudinal co-ordinate taken positive seawards, y the lateral co-ordinate,

and z the vertical co-ordinate, positive downwards. Let (u_x, u_y, u_z) and $(\Omega_x, \Omega_y, \Omega_z)$ be the velocity components and the components of the earth's rotation at any point in the estuary.

The three components of the equation of motion are

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\,[\Omega_y u_z - \Omega_z u_y], \qquad (3.1)$$

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\,[\Omega_z u_x - \Omega_x u_z], \qquad (3.2)$$

$$\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - 2[\Omega_x u_y - \Omega_y u_x], \qquad (3.3)$$

where p is the pressure and ρ the density. The viscous stress terms are not included in these equations since it is known that they are smaller than the turbulent stress terms by several orders of magnitude (Cameron and Pritchard (6), p. 319) and they may be regarded as being included in the Reynolds stresses in later equations.

Vertical accelerations may be neglected since they are typically in the order of 10^{-7} ms^{-2} whereas g and $\frac{1}{\rho}\frac{\partial p}{\partial z}$ are approximately 10 ms^{-2}, hence equation (3.3) may be replaced by

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \qquad (3.4)$$

We shall express the velocity vector $\underline{u}$ in the following form

$$\underline{u} = \underline{\bar{u}} + \underline{U} + \underline{u}',$$

where $\underline{\bar{u}}$ = the time mean velocity averaged over one or more tidal cycles,

$\underline{U}$ = the tidal velocity, which is assumed to be the sum of simple harmonic functions of the tidal period U_x, U_y, U_z, with amplitudes U_{ox}, U_{oy}, U_{oz} in directions x, y, z respectively.

It has been shown by a number of workers, in particular Johns (11), that the oscillating tidal water level, in addition to causing an oscillating current, gives rise to a small residual current. This residual current is included in $\underline{\bar{u}}$ so that strictly $\underline{U}$ is the oscillatory part of the tidal velocity.

$\underline{u}'$ = the turbulent velocity fluctuation which are assumed to have a time scale significantly smaller than the tidal period.

The equations of continuity for the turbulent, tidal and mean velocities are

$$\frac{\partial u'_x}{\partial x} + \frac{\partial u'_y}{\partial y} + \frac{\partial u'_z}{\partial z} = 0, \qquad (3.5)$$

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0, \qquad (3.6)$$

$$\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} = 0. \qquad (3.7)$$

assuming water to be incompressible.

The time mean equations are obtained from equations (3.1) and (3.2) by substituting for $\underline{u}$ and using equations (3.5), (3.6) and (3.7). In the operation of taking the time mean, terms of the type $<\bar{u}_x\ u'_y>$ and $<\bar{u}_x\ U_y>$ where <> indicates the time mean, are set equal to zero since by definition $<\bar{u}_x\ u'_y> = \bar{u}_x <u'_y>$ and $<u'_y> = 0$ and similarly $<U_y> = 0$. It is assumed that terms of the type $<U_x\ u'_y>$ are equal to zero since there is no reason to suspect a correlation between the oscillating tidal motion and the turbulent velocity fluctuations, although the mean square magnitude of the fluctuations will be greatest during the period when U_x is largest.

After averaging, the Coriolis forces appearing in equation (3.1) may be neglected since $\bar{u}_z$ is extremely small and $\bar{u}_y$ and U_y are assumed to be zero because of the chosen shape of the estuary (Pritchard (2)). The mean equations of motion are then given by

$$\frac{\partial \bar{u}_x}{\partial t} + \bar{u}_x \frac{\partial \bar{u}_x}{\partial x} + \bar{u}_z \frac{\partial \bar{u}_x}{\partial z} + \frac{\partial}{\partial x} <U_x\ U_x> + \frac{\partial}{\partial z} <U_x\ U_z> = - <\frac{1}{\rho} \frac{\partial p}{\partial x}>$$

$$- \frac{\partial}{\partial x} <u'_x\ u'_x> - \frac{\partial}{\partial y} <u'_y\ u'_x> - \frac{\partial}{\partial z} <u'_z\ u'_x>, \qquad (3.8)$$

and

$$0 = - <\frac{1}{\rho} \frac{\partial p}{\partial y}> - 2\ \Omega_z\ \bar{u}_x - \frac{\partial}{\partial x} <u'_x\ u'_y> - \frac{\partial}{\partial y} <u'_y\ u'_y> - \frac{\partial}{\partial z} <u'_z\ u'_y>. \qquad (3.9)$$

Equation (3.8) contains two terms involving the tidal components of the velocity. Since U_x has been assumed to be a simple harmonic function the first term is equivalent to $\frac{\partial}{\partial x}\langle U_x U_x\rangle = U_{ox}\frac{\partial U_{ox}}{\partial x}$. If, as is frequently the case, the tidal wave is a progressive wave, U_x and U_z are 90° out of phase, and the term $\frac{\partial}{\partial z}\langle U_x U_z\rangle$ disappears. A more thorough discussion of this point is given in Appendix 1; the present line of reasoning has been adopted to obtain equations similar to those of the previous workers. With this assumption equation (3.8) becomes:-

$$\frac{\partial \bar{u}_x}{\partial t} + \bar{u}_x \frac{\partial \bar{u}_x}{\partial x} + \bar{u}_z \frac{\partial \bar{u}_x}{\partial z} + U_{ox}\frac{\partial U_{ox}}{\partial x} = -\langle \frac{1}{\rho}\frac{\partial p}{\partial x}\rangle - \frac{\partial}{\partial x}\langle u'_x u'_x\rangle$$

$$-\frac{\partial}{\partial y}\langle u'_y u'_x\rangle - \frac{\partial}{\partial z}\langle u'_z u'_x\rangle. \qquad (3.10)$$

We shall now formulate the equation for salt continuity. If we neglect molecular diffusion on grounds that the turbulent diffusion is larger by several orders of magnitude, we have

$$\frac{\partial S}{\partial t} = -\frac{\partial(u_x S)}{\partial x} - \frac{\partial(u_y S)}{\partial y} - \frac{\partial(u_z S)}{\partial z}, \qquad (3.11)$$

where S is the salinity.

We assume that the salinity can be expressed in the form

$$S = \bar{S} + S_T + S' \qquad (3.12)$$

where $\bar{S}$ = the mean salinity,

S_T = the periodic variation in salinity due to the tidal motion, which is assumed to be simple harmonic with amplitude S_{To},

S' = the turbulent sluctuation in the salinity.

We can now substitute equation (3.12) into equation (3.11) and by taking time average over several tidal cycles obtain the mean salinity equation. The types of terms in this equation which can be set equal to zero are $\frac{\partial}{\partial x}\langle U_x S_T\rangle$, since U_x and S_T are usually 90° out of phase (Bowden (12), p. 27), $\frac{\partial}{\partial x}\langle U_x S'\rangle$ and $\frac{\partial}{\partial x}\langle S_T u'_x\rangle$, since there is no reason to believe that there is any correlation between the factors of these products (Cameron and Pritchard (6), p. 322). Hence equation (3.11) becomes

$$\langle\frac{\partial S}{\partial t}\rangle \equiv \frac{\partial \bar{S}}{\partial t} = -\frac{\partial}{\partial x}(\bar{u}_x \bar{S}) - \frac{\partial}{\partial y}(\bar{u}_y \bar{S}) - \frac{\partial}{\partial z}(\bar{u}_z \bar{S})$$

$$-\frac{\partial}{\partial x}\langle u'_x S'\rangle - \frac{\partial}{\partial y}\langle u'_y S'\rangle - \frac{\partial}{\partial z}\langle u'_z S'\rangle. \qquad (3.13)$$

4. A Two-Dimensional Model of Estuarine Flow

In (1) Pritchard wrote the salinity equation for the two-dimensional flow in an estuary in which the breadth B varies with x and z in the form:

$$\frac{\partial \bar{S}}{\partial t} = -\bar{u}_x \frac{\partial \bar{S}}{\partial x} - \bar{u}_z \frac{\partial \bar{S}}{\partial z} - \frac{1}{B}\frac{\partial}{\partial x}B\langle u'_x S'\rangle - \frac{1}{B}\frac{\partial}{\partial z}B\langle u'_z S'\rangle. \qquad (4.1)$$

In order to obtain this equation he assumed none of the variables appearing were dependent upon y. He presented a more rigorous derivation of an equivalent equation in (13).

The equations of motion, (3.9) and (3.10), and the two-dimensional equation of salt continuity (4.1) describe the flow in a stratified elongated estuary. Pritchard (1), (2) attempted to evaluate the relative importance of the different terms in these equations by using salinity and velocity data collected in the James River estuary in the summer of 1950 (Pritchard (14)). The general picture of salinity distribution in the James River is similar to that observed in many estuaries including the Thames (Inglis and Allen (15)), the Rotterdam Waterway (Rijkswaterstaat (16)) and Newcastle Harbour (Hinwood (17)).

The values of the terms in equation (4.1), the salt continuity equation, at different depths at one station are given in Table 1. From this table we see that the horizontal convective term $\bar{u}_x \frac{\partial \bar{S}}{\partial x}$ and the vertical turbulent term $\frac{1}{B}\frac{\partial}{\partial z}(B < u'_z S' >)$ are the dominant ones except near z = 3 m where $\bar{u}_x$ reverses direction and the vertical convective term $\bar{u}_z \frac{\partial \bar{S}}{\partial z}$ becomes important also. The time rate of change and the horizontal turbulent term are relatively small. Dyer (personal communication) has pointed out that the evaluation of the two turbulence terms given in Table 1 requires the neglect of one term then the neglect of the other, however the results are believed to give correct orders of magnitude for each term.

Table 1

Terms in the salt continuity equation, after Pritchard (1), expressed in $gm^{-3}\ s^{-1} \times 10^4$

Depth meters	$\frac{\partial \bar{S}}{\partial t}$	$\bar{u}_x \frac{\partial \bar{S}}{\partial x}$	$\bar{u}_z \frac{\partial \bar{S}}{\partial z}$	$\frac{1}{B}\frac{\partial}{\partial x}$ $B < u'_x S' >$	$\frac{1}{B}\frac{\partial}{\partial z}$ $B < u'_z S' >$
0.0	- 4.2	484.0	0.0	1.3	- 481.1
0.5	- 3.6	407.0	- 1.0	1.3	- 403.7
1.0	- 1.0	326.0	- 1.4	1.3	- 324.9
1.5	0.0	238.0	- 1.9	0.6	- 236.7
2.0	- 0.1	159.0	- 4.2	0.2	- 154.9
2.5	0.1	88.0	- 21.9	- 1.0	- 65.2
3.0	0.8	- 8.0	- 117.0	- 0.8	125.0
3.5	2.5	- 118.0	- 154.8	- 0.6	270.9
4.0	4.8	- 219.0	- 81.0	- 0.6	295.8
4.5	15.5	- 279.0	- 33.8	- 0.6	297.9
5.0	14.5	- 288.0	- 13.7	- 0.4	287.6
5.5	10.0	- 278.0	- 9.2	- 0.7	277.9
6.0	10.5	- 278.0	- 6.7	- 1.2	275.4
6.5	11.9	- 286.0	- 5.2	- 0.6	279.9
7.0	12.0	- 318.0	3.5	- 1.0	303.5
7.5	12.0	- 345.0	3.5	- 0.1	329.6

Let us now consider the equations of motion (3.9) and (3.10). In his 1956 paper (2), Pritchard argued that since the horizontal turbulent salt flux is small, as shown above, it would be reasonable to assume by analogy that the horizontal components of the turbulent flux of momentum are also negligible compared to the other terms of (3.9) and (3.10). Argument by analogy with salt flux appears to be weak because the above term in equation (4.1) may be small merely because u'_x and S' are not correlated, however the conclusions appear reasonable from the following line of argument.

The turbulent flux terms in equation (3.10) are $\frac{\partial}{\partial x} \langle u'_x u'_x \rangle + \frac{\partial}{\partial y} \langle u'_x u'_y \rangle + \frac{\partial}{\partial z} \langle u'_x u'_z \rangle$. Appropriate length scales for the changes in mean turbulence parameters will be the same as those for mean and tidal velocities, since these velocities give rise to the turbulence. Thus, since the fluctuating velocities are all of the same order of magnitude, the vertical term is likely to be greater than the longitudinal and transverse terms by factors of L/D and B/D respectively, where L is the length of the estuary. Very near the channel boundaries the turbulence will show pronounced anisotropy and this argument will not be correct, but it will be valid throughout most of the estuary. Thus equation (3.10) becomes

$$\bar{\bar{u}}_x \frac{\partial \bar{\bar{u}}_x}{\partial x} + \bar{\bar{u}}_z \frac{\partial \bar{\bar{u}}_x}{\partial z} + U_{ox} \frac{\partial U_{ox}}{\partial x} = - \langle \frac{1}{\rho} \frac{\partial p}{\partial x} \rangle - \frac{\partial}{\partial z} \langle u'_z u'_x \rangle. \tag{4.2}$$

Pritchard also assumed that equation (3.4) applied, and after averaging

he obtained the following:

$$\frac{1}{\rho} \frac{\partial \bar{\bar{p}}}{\partial z} - g = 0. \tag{4.3}$$

He then solved equations (4.2) and (4.3) for the shear stress term $\frac{\partial}{\partial z} \langle u'_z u'_x \rangle$. He made the physically reasonable assumptions that this stress was zero at the surface and could be estimated at the bottom from the Prandtl-von Karman boundary layer theory for a rough boundary with a roughness height of 0.02 cm. The shear stress could then be evaluated, and the relative importance of the different terms in (4.2) could be determined. The results of these calculations are given in Table 2.

Pritchard's assumption of 0.02 cm, although reasonable is quite critical and his shear stress values could easily be 50% or more in error. Assuming that he calculated the bed shear stress from a mean valocity measured 0.5 m above the bed, then a roughness height of 0.13 cm, which he cites as appropriate for a sandy bed, results in the shear stress being increased by a factor of 5½, and alters the computed horizontal pressure gradients.

Table 2

Variation with depth of the terms in the longitudinal component of the mean equation of motion for a typical section in the James River, after Pritchard (2), expressed in m s^{-2} x 10^6

Depth meters	$\bar{u}_x \frac{\partial \bar{u}_x}{\partial x}$	$\bar{u}_z \frac{\partial \bar{u}_x}{\partial z}$	$U_{ox} \frac{\partial U_{ox}}{\partial x}$	$-\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \rangle$	$\frac{\partial}{\partial z} \langle u'_z u'_x \rangle$
0	0.03	0.01	2.97	14.30	11.29
1	0.12	0.09	2.97	10.56	7.38
2	0.08	0.28	2.97	6.83	3.50
3	0.01	0.42	2.97	3.10	- 0.30
4	0.09	0.20	2.97	- 0.64	- 3.90
5	0.11	0.04	2.97	- 4.38	- 7.50
6	- 0.04	0.01	2.97	- 8.09	- 11.03
7	- 0.04	0.02	2.97	- 11.77	- 14.72

The convective terms in equation (4.2) are shown in table 2 to be negligible in comparison with the other terms. Hence equation (4.2) may be written as

$$U_{ox} \frac{\partial U_{ox}}{\partial x} = - \langle \frac{1}{\rho} \frac{\partial p}{\partial x} \rangle - \frac{\partial}{\partial z} \langle u'_z \; u'_x \rangle . \tag{4.4}$$

If the time-mean variables and parameters are assumed not to vary across the flow, the point equations (4.1) and (4.4) will

be unchanged on integration across the breadth of the estuary and equation (3.7) will become

$$\frac{\partial}{\partial x} B\bar{u}_x + \frac{\partial}{\partial z} B\bar{u}_z = 0. \tag{4.5}$$

Thus Pritchard's equations (4.1) and (4.4), together with the continuity equation (4.5) and appropriate boundary conditions form a mathematical model of an elongated estuary of gradually varying breadth in which the variables and parameters are assumed not to vary across the breadth. Two more relationships are required. The first is an equation of state which relates ρ and S. For the type of estuary we are considering the following expression can be used:

$$\rho = \rho_f \,(1 + kS) \tag{4.6}$$

where ρ_f is the density of fresh water and k is a constant. The second is a relationship between the mean and the turbulence quantities such as the eddy coefficients introduced in the next section.

5. Development of the model

In order to treat the model developed in the previous section, it is necessary to simplify it. It is usual to relate the shear stress to the mean velocity gradient by a coefficient of eddy viscosity, A, defined by

$$\langle u'_z \; u'_x \rangle = - A \frac{\partial \bar{u}_x}{\partial z} . \tag{5.1}$$

Thus equation (4.4) becomes

$$U_{ox} \frac{\partial U_{ox}}{\partial x} = - \left\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \right\rangle + \frac{\partial}{\partial z} \left(A \frac{\partial \bar{u}_x}{\partial z}\right). \qquad (5.2)$$

Similarly the turbulent fluxes of salt are related to the mean salinity gradients by coefficients of eddy diffusion, K_x and K_z, defined by

$$\langle u_x' S' \rangle = - K_x \frac{\partial \bar{S}}{\partial x}, \qquad (5.3)$$

$$\langle u_z' S' \rangle = - K_z \frac{\partial \bar{S}}{\partial z}. \qquad (5.4)$$

In general the eddy coefficients are not constants, but are functions of position and the parameters of the flow taking appropriate values at each point. For steady conditions equation (4.1) becomes

$$\bar{u}_x \frac{\partial \bar{S}}{\partial x} + \bar{u}_z \frac{\partial \bar{S}}{\partial z} = \frac{1}{B} \frac{\partial}{\partial x} \left(B K_x \frac{\partial \bar{S}}{\partial x}\right) + \frac{1}{B} \frac{\partial}{\partial z} \left(B K_z \frac{\partial \bar{S}}{\partial z}\right). \qquad (5.5)$$

From the results in Table 1 we can deduce that the term containing K_x must usually be fairly small compared to the term containing K_z.

Further simplifications were obtained by Hansen and Rattray who reduced the equation of motion to a linear equation. A more rigorous reduction is given below. The resulting equation is partly obtained by differentiation of the simplified equations (4.3) and (5.2), and it is by no means certain that the derivatives of the terms neglected in obtaining the latter equations are also negligible. A more

rigorous procedure is to consider the order of magnitude of terms after all differentiation; this will be discussed in a later paper in this series.

The first such simplification is to remove the pressure from equations (4.3) and (5.2). The difficulty in doing this is that the pressure appears in (4.3) in the form $\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x}$ while in (5.2) it appears in the form $\left\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \right\rangle$. By assuming that there is no correlation between ρ' and p' or from the arguments presented in Appendix 2 it may be shown with sufficient accuracy, that

$$\left\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \right\rangle = \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x}, \qquad (5.6)$$

which is implicitly assumed in the three papers by Hansen and Rattray.

The equation (4.5) can be satisfied by a stream-function ψ defined by $\frac{\partial \psi}{\partial z} = - B \bar{u}_x$, $\frac{\partial \psi}{\partial x} = B \bar{u}_z$. If p and ρ are eliminated from equations (4.3) and (5.2) using (5.6) and the equation of state (4.6) we obtain after some manipulation and by using the fact that the tidal term is independent of z (Table 2):

$$U_{ox} \frac{\partial U_{ox}}{\partial x} \; k \frac{\partial \bar{S}}{\partial z} + \frac{\partial^2}{\partial z^2} \left[A \frac{\partial}{\partial z} \left(\frac{1}{B} \frac{\partial \psi}{\partial z}\right)\right]$$

$$+ k \frac{\partial}{\partial z} \left\{\bar{S} \frac{\partial}{\partial z} \left[A \frac{\partial}{\partial z} \left(\frac{1}{B} \frac{\partial \psi}{\partial z}\right)\right]\right\} + g k \frac{\partial \bar{S}}{\partial x} = 0. \qquad (5.7)$$

This equation can be simplified if certain conditions are satisfied. In order to obtain these conditions we shall nondimensionalize (5.7) by defining new variables as follows:

$$\psi = \psi_o \psi^*, \quad \overline{S} = S_o S^*, \quad z = L_1 \eta_1,$$

$$x = L_o \xi, \; z = L_2 \eta_2.$$

Here ψ_o and S_o are reference values of ψ and S, L_o the length scale for horizontal variation in $\overline{S}$, L_1 the length scale for vertical variation in ψ, and L_2 the length scale for vertical variation in $\overline{S}$. It seems reasonable to suppose that L_1 is of the same order of magnitude as the depth of the estuary, while L_2 may possibly be very much smaller if there is a thin layer in which the salinity varies rapidly. We shall also write

$$A = A_o A^*, \; B = B_o B^*,$$

where A_o and B_o are reference values of A and B.

In terms of these variables and parameters equation (5.7) becomes

$$\frac{\partial^2}{\partial \eta_1^2} [A^* \frac{\partial}{\partial \eta_1} (\frac{1}{B^*} \frac{\partial \psi^*}{\partial \eta_1})] + k\, S_o\, S^* \frac{\partial^2}{\partial \eta_1^2} [A^* \frac{\partial}{\partial \eta_1} (\frac{1}{B^*} \frac{\partial \psi^*}{\partial \eta_1})]$$

$$+ k\, S_o \frac{L_1}{L_2} \frac{\partial S^*}{\partial \eta_2} \frac{\partial}{\partial \eta_1} [A^* \frac{\partial}{\partial \eta_1} (\frac{1}{B^*} \frac{\partial \psi^*}{\partial \eta_1})]$$

$$+ U_{ox} \frac{\partial U_{ox}}{\partial x} \frac{B_o L_1^4\, kS_o}{A_o L_2 \psi_o} \frac{\partial S^*}{\partial \eta_2} + \frac{g\, k\, S_o\, B_o\, L_1^4}{\psi_o\, A_o\, L_o} \frac{\partial S^*}{\partial \xi} = 0. \qquad (5.8)$$

In comparison with the first term, the second term is of lower order of magnitude since $kS_o << 1$. The suppression of this term can be called a Boussinesq type approximation. The third term, however, requires more discussion. This term can only be ignored if L_1/L_2 is of order 1, but if there are parts of the estuary in which the vertical salinity gradient varies relatively rapidly over a length small compared L_1 the ratio L_1/L_2 would be large. Hence $kS_o\ (L_1/L_2)$ could be of order 1, and in this case the third term would have to be retained. To ensure that $k\, S_o(L_1/L_2)<<1$ this paper is restricted to slightly stratified estuaries, i.e. with no strong salinity gradients at any point. The restriction of small salinity gradients allows us to omit the product of the tidal acceleration and the vertical salinity gradient in equation (5.8). The restriction imposed is that

$$U_{ox} \frac{\partial U_{ox}}{\partial x} \frac{B_o L_1^4\ kS_o}{A_o L_2 \psi_o} << 1$$

or an equivalent but more complete expression obtainable from the results of Appendix 1.

Thus in a slightly stratified estuary equation (5.8) may be reduced to

$$\frac{\partial^2}{\partial z^2} [A \frac{\partial}{\partial z} (\frac{1}{B} \frac{\partial \psi}{\partial z})] + g\, k \frac{\partial \overline{S}}{\partial x} = 0. \qquad (5.9)$$

In terms of the stream function ψ, the salt continuity equation (5.5) becomes

$$\frac{\partial\psi}{\partial x}\frac{\partial\bar{S}}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\bar{S}}{\partial x} = \frac{\partial}{\partial x}(B\,K_x\frac{\partial\bar{S}}{\partial x}) + \frac{\partial}{\partial z}(B\,K_z\frac{\partial\bar{S}}{\partial z}). \qquad (5.10)$$

The velocity components and the salinity must satisfy certain conditions at the boundaries. Only the conditions at the surface and the bottom will be given here; these conditions are

1) the net transport is equal to the river flow R,
2) the shear stress at the surface is equal to the wind stress,
3) there is no normal flow and no slip at the bottom $z = D(x)$,
4) the normal salt flux is zero at the surface and the bottom, or equivalently, the salt flux is zero across any section and is zero at either the surface or the bottom.

Thus we have

$$\psi = R,\ \frac{\partial}{\partial z}(\frac{1}{B}\frac{\partial\psi}{\partial z}) = \frac{\tau}{\rho_f A} \quad \text{at } z = 0, \qquad (5.11a)$$

$$\psi = -\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial x}\frac{dD}{dx} = 0 \quad \text{at } z = D(x), \qquad (5.11b)$$

and any two of

$$K_z\frac{\partial\bar{S}}{\partial z} - K_x\frac{\partial\bar{S}}{\partial x}\frac{dD}{dx} = 0 \quad \text{at } z = D(x), \qquad (5.12a)$$

$$B\,K_z\frac{\partial\bar{S}}{\partial z} = 0 \quad \text{at } z = 0, \qquad (5.12b)$$

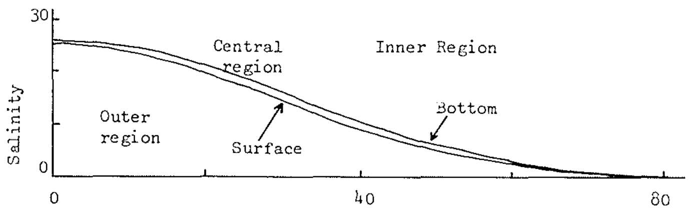
$$\int_0^D (\bar{S}\frac{\partial\psi}{\partial z} + B\,K_z\frac{\partial\bar{S}}{\partial x})\,dz = 0. \qquad (5.12c)$$

It must be pointed out that since the viscous terms no longer appear explicitly in equation (5.9) the use of the no slip condition restricts the choice of eddy viscosities and may introduce difficulties in the analysis.

6. Similarity solutions

Similarity solutions of the model derived in the previous section have been obtained by Rattray and Hansen (3), Hansen (4), and Hansen and Rattray (5). Before we discuss these solutions it is necessary to remark that although they attribute their longitudinal equation of motion to Pritchard they write an equation similar to (5.2) differently in each paper. These differences are in the definition of the eddy viscosity and in the form of the tidal term, and necessitate various assumptions in order to arrive at equation (5.9).

Based on Pritchard's data, Hansen and Rattray (5) divided the estuary into the three regions shown in Figure 2. The other data cited in Section 4 show the same trends, and although the regions merge into each other such a division appears to be reasonable. It is to be regretted that this expedient is necessary because it has the effect of introducinganother parameter whose magnitude is to be determined from the measurements thereby reducing the value of the comparison of theory and measurement.



Distance from mouth
(Values given are 80-x where x is the longitudinal co-ordinate)

FIGURE 2 LONGITUDINAL SALINITY DISTRIBUTION IN THE DELAWARE RIVER FROM PRITCHARD (HANSEN AND RATTRAY (5), VALUES GIVEN ARE 80 - x).

In their treatment of the outer region, Rattray and Hansen (3) assumed that

1) K_x is zero,
2) A, K_z, and B are constant,
3) the net transport R is only a small part of the total circulation, so it can be ignored to a first order
4) the surface salinity is prescribed
5) vertical advective salt flux is zero.

It is evident that assumption (3) is reasonable as far as momentum and water fluxes in this region are concerned but in order to create proper salinity distribution it necessitates an assumption such as (4) above. These assumptions were justified by reference to

the data of Pritchard, although Table 1 shows that assumption (5) may not be justified.

If a new variable, $\sigma = g\ (1 - \frac{\bar{S}}{S_o})$, is introduced, where S_o is the salinity of sea water, equations (5.9) and (5.10) become

$$\frac{A}{B}\frac{\partial^4\psi}{\partial z^4} - \varepsilon\frac{\partial\sigma}{\partial x} = 0, \qquad (6.1)$$

$$K_z\frac{\partial^2\sigma}{\partial z^2} + \frac{1}{B}\frac{\partial\psi}{\partial z}\frac{\partial\sigma}{\partial x} = 0, \qquad (6.2)$$

where ε denotes the differential specific gravity kS_o. This parameter is small compared to unity. The boundary conditions become

$$\psi = \frac{\partial\sigma}{\partial z} = 0,\ \frac{1}{B}\frac{\partial^2\psi}{\partial z^2} = \frac{\tau}{A\rho_f},\quad \text{at } z = 0, \qquad (6.3a)$$

and

$$\psi = \frac{\partial\sigma}{\partial z} = \frac{\partial\psi}{\partial z} = 0,\quad \text{at } z = D\ (x) \qquad (6.3b)$$

although comparison with equations (5.11) and (5.12) shows that D (x) is now being assumed constant.

The similarity transformation used by Rattray and Hansen is

$$\psi\ (x,z)/B = K_z\ \xi^{\alpha+1}\ F(\eta),\ \sigma(x,z) = A\ K_z L_o^{-3}\xi^{5\alpha+1}G(\eta),$$

where $\xi = x/L_o$, and L_o and α are chosen so that the dimensionless variable $\eta = z\, L_o^{-1}\, \xi^{\alpha}$ varies from zero at the surface to unity at the bottom of the estuary, subject to the restriction that α cannot be greater than -0.4 in order to satisfy the realistic estuarine condition of increasing depth and approach of salinity to that of the ocean. This implies that the depth is $D = L_o^{1+\alpha}\, x^{-\alpha}$, and thus the boundary condition (6.3b) no longer corresponds to no slip at the bottom.

The equations for F and G are

$$F'''' - \varepsilon[(5\alpha+2)G + \alpha\eta G'] = 0,$$

$$G'' + F'[(5\alpha+2)G + \alpha\eta G'] = 0.$$

Approximate solutions were obtained by expanding F and G in powers of ε. The details of the solution and the comparison of it with field observations can be found in Rattray and Hansen (3). However, since the boundary condition (6.3b) can no longer be given a physical interpretation, it is difficult to estimate the validity of the solution.

In Hansen and Rattray (5) the central and the inner regions are considered. In formulating their equations for the central region they assume that D, A and K_z do not vary along the estuary and that K_x, which is non-zero, increases seaward at a rate proportional to the river discharge velocity, i.e.

$$\frac{dK_x}{dx} = \frac{R}{BD}.$$

For mathematical simplicity they then assumed that the width, B, and all the turbulent exchange coefficients do not vary with depth and that the river discharge, R, is constant indicating that no tributary inflows occur.

Solutions of equations (5.9) and (5.10) are sought in the form

$$\psi\,(x,z) = R\,\phi\,(\eta), \quad S(x,z) = S_o\,[\nu\,\zeta + \theta(\eta)]$$

where ν is a constant, $\eta = z/D$, $\zeta = R(x-x_o)/BDK_{xo}$, and $x = x_o$ is any chosen reference section at which the average value of S is S_o. Equations (5.9) and (5.10) become

$$\phi'''' + \nu R\alpha = 0, \; M\,\theta'' + \nu(\phi' + 1) = 0,$$

with boundary conditions

$$\phi(1) = \phi'(1) = 0, \quad \phi(0) = 1, \quad \phi''(0) = T,$$

and any two of

$$\theta'\,(0) = 0,$$

$$\theta'\,(1) = 0,$$

$$\nu + \int_0^1 \phi'\,\theta d\,\eta = 0.$$

This boundary problem is characterized by the three dimensionless parameters.

$$T = \frac{B\ D^2\ \tau}{AR},$$

$$Ra = \frac{g\ k\ S_o\ D^3}{AK_{xo}},$$

$$M = \frac{K_z\ K_{xo}\ B^2}{R^2},$$

where K_{xo} is a reference value of K_x. Hansen and Rattray interpret these as the dimensionless wind stress, the estuarine analog of the Rayleigh number, and a ratio of tidal mixing to river flow, respectively. The final parameter could be better regarded as the square of this ratio, provided that $\sqrt{K_z\ K_{xo}}$ is taken to be a typical tidal mixing coefficient.

The solutions are:

$$\phi\ (\eta) = 1/2(2-3\eta + \eta^3) - 1/4\ T\ (\eta - 2\ \eta^2 + \eta^3) - \frac{\nu Ra}{48}\ (\eta - 3\eta^3 + 2\eta^4),$$

$$\frac{\bar{S}}{S_o} = 1 + \nu\zeta + \frac{\nu}{M}\ \left[(\eta - 1/2) - 1/2(\eta^2 - \tfrac{1}{3}) - \int_0^\eta \phi d\eta + \int_0^1 \int_0^\eta \phi d\eta' d\eta\right].$$

For appropriate values of T, ν, Ra, and M these solutions agree fairly well with observations. The detailed comparison is given in the paper referred to. It is interesting to note that it is necessary to take K_x to be a non-zero function of x in order to obtain this solution. Hansen (4) had previously obtained the same solutions

and also a non-similarity solution to the boundary value problem defined in Section 5.

Hansen and Rattray also obtained a solution for the inner region by using a procedure similar to the one used in their treatment of the outer region. The resultant similarity equations are solved approximately by expanding the dependent variables in power series in $\frac{1}{M}$. This procedure is justified, they state, for estuaries with strong tidal currents. Again the details can be found in Hansen and Rattray (5).

7. Conclusions

Pritchard's model of a coastal plain estuary has been shown to be a satisfactory basis for further studies provided that the assumptions discussed above are satisfied. Pritchard's model has been developed by Rattray and Hansen who have made additional assumptions and simplifications in order to obtain their solutions.

An assumption, not made explicitly by these authors, is that the vertical salinity gradients should be very small everywhere, i.e. the estuary is slightly stratified. This may be taken in conjunction with their assumptions about the geometry to define the class of estuary under consideration. An unacceptable assumption is the neglect of river flow by Rattray and Hansen (3).

The solutions of the model depend strongly on the choice of the forms that the turbulent transfer coefficients A, K_x, K_z are assumed to have. The concept of eddy transfer coefficients does not rest on a firm theoretical basis, and the validity of their use with time mean velocity gradients, which ignore tidal currents, is questionable. However, it seems that at the present time a quantitative treatment of estuarine flow can only be attempted if these coefficients are used and are assumed to be constant or to vary in a simple manner. Thus one application of this model is to examine various possible forms of A, K_x and K_z to see which assumptions yield physically realistic solutions. Using very simple assumptions for the mode of vairation of A, K_x and K_z, Rattray and Hansen have obtained quite good agreement between their solutions and Pritchard's data. Unfortunately, as is characteristic of such problems, there is a large number of parameters whose values are subject to considerable uncertainty, thus weakening the reliability of this confirmation.

In practice it appears that solutions of the model could be applied to fill in gaps or extend the range of field measurements and to provide a useful approximation for checking against numerical solutions of more sophisticated models which should be based on a clear understanding of all the underlying assumptions and approximations.

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Appendix 1 - Tidal Terms

Suppose $U_x = [U_{1x} \cos\omega(x-ct) + U_{2x} \cos\omega(x+ct)]\, F_x (z)$,

$$U_z = [U_{1z} \cos(\omega(x-ct) + \varepsilon) + U_{2z} \cos(\omega(x+ct)+\varepsilon)]\, F_z (z)$$

which permits a progressive, standing or intermediate tide wave of small amplitude in a uniform channel withour friction (the notation for velocities and co-ordinates is the same as in the text, but other symbols apply only to this appendix).

The continuity equation is

$$0 = \frac{\partial U_x}{\partial x} + \frac{\partial U_z}{\partial z} = -k\, [U_{1x} \sin k(x-ct) + U_{2x} \sin k(x+ct)]\, F_x (z)$$

$$+ [U_{1z} \cos(k(x-ct)+\varepsilon) + U_{2z} \cos(k(x+ct)+\varepsilon)]\, F_z' (z),$$

hence $\frac{F_x}{F_z'} = \frac{U_{1z} \cos(k(x-ct)+\varepsilon) + U_{2z} \cos(k(x+ct)+\varepsilon)}{k[U_{1x} \sin k(x-ct) + U_{2x} \sin k(k+ct)]} = K$

where K is a finite constant and the primes denote differentiation.

If $x = t = 0$ the second of these equations can only a apply if $\varepsilon = \pm \frac{\Pi}{2}$. Taking $\varepsilon = -\frac{\Pi}{2}$

$$(U_{1z} - kKU_{1x}) \sin k(x-ct) + (U_{2z} - kK U_{2x}) \sin k(x+ct) \equiv 0.$$

Therefore, $U_{1z} = kKU_{1x}$, $U_{2z} = kKU_{2x}$;

hence $U_z = kK[U_{1x} \sin k(x-ct) + U_{2x} \sin k(x+ct)]F_z (z)$

and $<U_x U_z> = \frac{1}{T} \int_t^{t+T} U_x U_z \, dt$

$$= U_{1x} U_{2x} kK F_x F_z \sin 2kx.$$

Similarly $<U_x U_x> = F_x^2 (\tfrac{1}{2}(U_{1x} + U_{2x})^2 + 2 U_{1x} U_{2x} k \sin^2 kx)$.

In the equation of motion the tidal terms appear as derivatives and in the final equation (eqn. 5.8) they should appear as

$$\frac{\partial^2 <U_x U_z>}{\partial z^2} = U_{1x} U_{2x} kK (F_x'' F_z + 2F_x' F_z + F_x F_z'')\sin 2kx$$

and $\frac{\partial^2 <U_x U_x>}{\partial x \, \partial z} = 4 k U_{1x} U_{2x} F_x \sin 2kx.$

Both of these terms disappear if the tide is a pure progressive wave (i.e. if $U_{2x} = 0$), if there is no significant variation of tidal velocity with depth (F_x constant), or if the length of the tidal wave $\frac{2\Pi}{k}$ is much greater than that of the estuary. In all other cases both tidal terms must be retained.

Appendix 2 - The Density-Pressure Product Term

The complete pressure term in the equation of motion is, in vector form.

$$- <\nabla x (\frac{1}{\rho} \nabla p)> = <\nabla p \; x \nabla \frac{1}{\rho}> = <\nabla \bar{p} \; x \; \nabla \frac{1}{\bar{\rho}}> + <\nabla \frac{\rho'}{(\bar{\rho}+\rho')\bar{\rho}} \; x \; \nabla p'>.$$

The first term on the right hand side has already been considered, the second term involves the product of derivatives of density and pressure fluctuations. In the y direction the latter term becomes:

$$< \nabla \frac{\rho'}{(\bar{\rho} + \rho')\bar{\rho}} \times \nabla p' > \cdot \hat{\jmath} = \frac{\partial}{\partial x} \frac{\rho'}{(\bar{\rho} + \rho')\bar{\rho}} \frac{\partial p'}{\partial z} > - <\frac{\partial}{\partial z} \frac{\rho'}{(\bar{\rho} + \rho')\bar{\rho}} \frac{\partial p'}{\partial x} >$$

$$= \frac{1}{\rho^2} < \frac{\partial \rho'}{\partial x} \frac{\partial p'}{\partial z} - \frac{\partial \rho'}{\partial z} \frac{\partial p'}{\partial x} >$$

plus term of higher order in $\rho'/\bar{\rho}$.

Applying mixing lengths ideas, a density fluctuation may be regarded as the result of a circular eddy acting mean density gradients, hence typical values for a slightly stratified estuary are

$$\frac{\partial \rho'}{\partial z} \sim \frac{\partial \rho'}{\partial x} \sim \frac{\partial \bar{\rho}}{\partial x} \sim \rho_o \; k \; S_o/L.$$

The pressure at a point will fluctuate if the average density of the water above it fluctuates. Assuming that an eddy of diameter up to D and density differential ρ' passes over the point in question, the pressure will fluctuate an amount $p' \leq \rho' g \; D$, and the gradients will have the values

$$\frac{\partial p'}{\partial x} \sim \frac{\partial p'}{\partial z} \sim \rho' g \sim g \rho_o \; k \; S_o/L.$$

Thus the fluctuation term is of the order of $g(\frac{k \; S_o}{L})^2$, whereas the

gravitational term in the equation of motion is of order $g(\frac{k \; S_o}{LD})$. Since both kS_o and D/L are small quantities the neglect of the fluctuation term appears to be reasonable without the need to assume that ρ' and p' are uncorrelated.

ON FLOW IN ESTUARIES
PART II: A SLIGHTLY STRATIFIED TURBULENT FLOW

I.- INTRODUCTION.

Flow in a two-dimensional channel of uniform depth is considered. The flow is assumed to be incompressible but non-homogeneous due to differences in any conservative property, although for convenience the particular example of a horizontal channel containing water of varying salinity will be described. Through the action of turbulence, vertical and horizontal mixing occur, and in addition convective fluxes of salt may occur.

An important example of this idealised flow is the coastal plain or slightly stratified estuary discussed by Rasmussen and Hinwood (1) - referred to as Part I - and by Pritchard (2), Hansen (3), Hansen and Rattray (4) and many others. River water enters the estuary at one end and sea water at the other end, predominantly at the bottom. Mixing occurs mainly over the central region of the estuary, and the salinity satisfies boundary conditions at both ends of the estuary.

In this paper we show that it is not possible for the flow to be two-dimensional and slightly stratified, the channel to be of constant depth and the turbulent diffusion to be expressed by eddy diffusion coefficients. Some or all of these conditions have been

assumed in previous analyses of slightly stratified estuaries, and even though in this paper we do not attempt to propose improved models, it was felt that a careful analysis of the limitations of the models presently in use would be of value. We will present an analysis of more general models in a later paper.

2. Formulation

The problem will be formulated and analysed in its simplest form for clarity; then some important generalizations will be treated. The turbulent incompressible flow in a channel of a non-homogeneous fluid is considered and it is assumed that the following conditions are satisfied:

(i) The time-mean velocity and salinity are two-dimensional.

(ii) The depth of the channel, D, is constant.

(iii) The flow is slightly stratified.

(iv) The diffusive mixing may be attributed to the effects of diffusion coefficients which are functions of position.

By the third condition we mean that the vertical salinity gradient is small at any point and that the salinity throughout the channel may be represented by a function of the longitudinal coordinate only plus a small deviation which is a function of both position coordinates.

We shall now show that these four conditions cannot all be satisfied simultaneously. We do this by assuming that they are satisfied, and then by integration of the salt conservation equation we obtain a contradiction. It is not necessary to make any detailed assumption about the dynamics of the flow. Both steady and unsteady flows are considered, but the boundary conditions are taken as steady in both cases for reasons given later.

Let x_1 be the horizontal coordinate measured along the channel with its origin so chosen that the mixing mainly takes place between $x_1 = -L$ and $x_1 = L$. We shall later discuss the different cases which arise from the different magnitudes of the ratio D/L. Let z_1 be the vertical coordinate taken positive downwards and with the origin at the surface.

At the ends of the channel we can suppose that the flow is uniform and that the derivative of the salinity with respect to x_1 tends to zero as $x_1 \to \pm\infty$. Since we are considering the mixing of fresh and salt water it can also be supposed that the salinity is not constant throughout the channel. This density difference causes an inflow of saltwater into the mixing region $|x_1| < L$, and in order to keep this region stationary it is necessary to suppose a forced inflow of freshwater at the rate R, say, in the direction of positive x_1. The stratification is then caused by the partial mixing due to the turbulence of these two streams of fluid.

The turbulent salt conservation equation, which was given in Part I, is

$$\frac{\partial \bar{S}}{\partial t_1} = \frac{\partial \psi}{\partial z_1}\frac{\partial \bar{S}}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \bar{S}}{\partial z_1} + \frac{\partial}{\partial x_1}\left(G(x_1,z_1,t_1)\frac{\partial \bar{S}}{\partial x_1}\right) + \frac{\partial}{\partial z_1}\left(H(x_1,z_1,t_1)\frac{\partial \bar{S}}{\partial z_1}\right). \tag{2.1}$$

Here the time mean salinity is $\bar{S}$ and the time mean stream function ψ identically satisfies the two-dimensional continuity equation. In obtaining this expression the following assumptions were made:

(1) Molecular diffusion is negligible compared with turbulent diffusion.

(2) The flux of salt due to turbulence may be represented by the product of the mean salinity gradient and an eddy diffusion coefficient, G, in the x direction and H in the z direction.

(3) All quantities are assumed to be independent of the transverse coordinate.

(4) Time means of products of tidal terms are zero as are time means of tidal and turbulence terms.

Boundary conditions at the surface and the bottom of the channel which ensure conservation of water and salt are

$$\text{at } z_1 = 0 \quad \psi = R, \quad \frac{\partial \bar{S}}{\partial z_1} = 0;$$

$$\text{at } z_1 = D \quad \psi = 0, \quad \frac{\partial \bar{S}}{\partial z_1} = 0, \tag{2.2}$$

where R is the net flux of water per unit breadth, i.e. the river discharge per unit breadth. At the ends of the channel we have the following conditions on $\bar{S}$

$$\frac{\partial \bar{S}}{\partial x_1} \to 0 \quad \text{as} \quad x_1 \to \pm\infty, \tag{2.3}$$

and that $\frac{\partial \bar{S}}{\partial x_1}$ is not identically zero throughout the channel. From the condition that the flow is uniform at the ends of the channel we have

$$\frac{\partial \psi}{\partial x_1} \to 0 \quad \text{as} \quad x_1 \to \pm\infty. \tag{2.4}$$

To normalize the problem consisting of (2.1) to (2.4) we at first assume that the significant length scales for changes in $\bar{S}$ and ψ are equal and later treat the case of unequal scales. Thus we define non-dimensional coordinates x and z and variables as follows:

$$x_1 = Lx,$$

$$z_1 = Dz,$$

$$t_1 = t_o t,$$

$$\psi(x_1,z_1,t_1) = \psi_o\ \phi(x,z,t),$$

$$\bar{S}(x_1,z_1,t_1) = S_o\ \theta(x,z,t),$$

$$G(x_1,z_1,t_1) = K_x\ g(x,z,t),$$

$$H(x_1,z_1,t_1) = K_z\ h(z,x,t)$$

where the subscript o indicates a suitable reference quantity and K_x and K_z are reference values of G and H. Equation (2.1) now becomes

$$T\frac{\partial \theta}{\partial t} = -\frac{\partial \phi}{\partial x}\frac{\partial \theta}{\partial z} + \frac{\partial \phi}{\partial z}\frac{\partial \theta}{\partial x} + X\frac{\partial}{\partial x}\left(g\frac{\partial \theta}{\partial x}\right) + Z\frac{\partial}{\partial z}\left(h\frac{\partial \theta}{\partial z}\right) \tag{2.5}$$

where $T = \frac{LD}{\psi_o t_o}$, $X = K_x D/L\psi_o$ and $Z = K_z L/D\psi_o$.

The assumption that the flow is slightly stratified now enables θ to be expanded in terms of a small parameter α

$$\theta(x,z,t) = f(x,t) + \sum_{n=1}^{\infty} \alpha^n\ \theta_n(x,z,t), \tag{2.6}$$

and the boundary conditions (2.2) and (2.3) are satisfied if

$$\frac{\partial \theta_n}{\partial z} = 0 \quad \text{at } z = 0,\ 1 \text{ for } n \geq 1, \tag{2.7}$$

$$f'(x,t) \to 0, \quad \frac{\partial \theta_n}{\partial x} \to 0 \text{ for } n \geq 1 \tag{2.8}$$

as $x \to \infty$, and

$$f'(x,t) \to 0, \quad \frac{\partial \theta_n}{\partial x} \to 0 \text{ for } n \geq 1 \tag{2.9}$$

as $x \to -\infty$. Also $f'(x,t)$ is not identically zero for $-\infty < x < \infty$.

The stream function ϕ may also be expanded in terms of α

$$\phi(x,z,t) = \phi_o(x,z,t) + \sum_{n=1}^{\infty} \alpha^n\ \phi_n(x,z,t) \tag{2.10}$$

with the boundary conditions (2.2) and (2.3) becoming

$$\phi_o = R/\psi_o \quad \text{at } z = 0, \quad \phi_o = 0 \text{ at } z = 1,$$

$$\phi_n = 0 \quad \text{at } z = 0, \quad 1 \text{ for } n > 1, \tag{2.11}$$

and

$$\frac{\partial \phi_n}{\partial x} \to 0 \quad \text{as } x \to \pm\infty \quad \text{for } n \geq 1. \tag{2.12}$$

The expansion (2.10) implies that the presence of slight vertical stratification causes an equally slight modification to the streamline patterns which is unlikely to be true unless α is extremely small. In section 4 this restriction will be removed.

In the next two sections steady flows are considered; in section 3 steady flows with equal length scales for salinity and stream function and in section 4 other steady flows. The latter comprise steady flows with unequal length scales for salinity and stream function, flows in which the small vertical stratification causes a large change in stream function and finally steady flows in a channel of gradually varied depth. In section 5 the restriction of steady flow is removed and two types of flow are considered: flows in which the salinity is steady to first order and flows in which the salinity is unsteady to first order. These examples include most of the cases occurring in estuarine flows; some of the cases omitted are considered in the final section.

3. Steady Flow with Equal Length Scales

The basis of the analysis is to assume that the four conditions stated in section 2 are satisfied and then by integration of the linearized salt conservation equation obtain a contradiction.

We substitute expansions (2.6) and (2.10) into the steady form of equation (2.5) and retain only terms of order α^o. At the moment we do not wish to make any assumptions about the magnitudes of X and Z except that both convective and diffusive terms should be present in the linearized approximate equation. Hence we have

$$-f'(x) \frac{\partial \phi_o}{\partial z} = X \frac{\partial g f'(x)}{\partial x} + \alpha Z \frac{\partial}{\partial z} \left(h \frac{\partial \theta_1}{\partial z}\right) \tag{3.1}$$

with boundary conditions given by (2.7) to (2.9), (2.11) and (2.12). From this equation we see that in this model of slightly stratified flow only horizontal convection of salt is retained. The vertical convection is of lower order.

Depending on the magnitudes of X and Z there are three non-trivial cases to be considered.

Case 1 $X = O(1)$, $Z = O(1)$;

Case 2 $X = O(1)$, $Z = O(\alpha^{-1})$;

Case 3 $X = O(\alpha)$, $Z = O(\alpha^{-1})$.

From the definitions of X and Z given in the previous section we have

$$\frac{X}{Z} = \frac{K_x}{K_z}\left(\frac{D}{L}\right)^2$$

Thus if we suppose that the ration K_x/K_z is of order 1, the first case implies that the depth of the channel and the length of the mixing region are of the same order of magnitude, while for the second and third cases the length of the mixing region is greater, by several order of magnitudes, than the depth of the channel. For application to the flow in slightly stratified estuaries the second case is undoubtedly the more realistic one. We obtain, however, the same result from the analysis of each of the three cases.

Case 1.

Equation (3.1) reduces to

$$-f'(x)\frac{\partial\phi_o}{\partial z} = X\frac{\partial}{\partial x}(gf'(x)). \tag{3.2}$$

When this equation is integrated with respect to z, we obtain

$$-f'(x)\phi_o(x,z) = X\int_0^z \frac{\partial}{\partial x}\left[g(x,z)f'(x)\right]dz + F_o(x).$$

Since from (2.11) $\phi_o = R/\psi_o$ at $z = 0$ and $\phi_o = 0$ at $z = 1$,

$$-\frac{R}{\psi_o}f'(x) = F_o(x),$$

and

$$0 = X\int_0^1 \frac{\partial}{\partial x}\left[g(x,z)f'(x)\right]dz + F_o(x).$$

By substituting for $F_o(x)$ from the first equation into the second we get

$$X\int_0^1 \frac{\partial}{\partial x}\left[g(x,z)f'(x)\right]dz - \frac{R}{\psi_o}f'(x) = 0. \tag{3.3}$$

The order of integration and differentiation in equation (3.3) may be interchanged following Leibnitz's rule provided that $g(x,z)$, $f'(x)$ and $\frac{\partial}{\partial x}g(x,z)f'(x)$ are continuous in both x and z for $-\infty < x < \infty$ and $0 \leq z \leq 1$. In flows of the type we are considering these conditions are met, and thus we may exchange the order of the operations and then integrate with respect to x to obtain

$$\int_0^1\left[g(x,z)f'(x)\right]dz - \frac{R}{X\psi_o}f(x) = C_o$$

where C_o is a constant. This equation may be written in the form

$$f'(x) - \alpha P(x)f(x) = C_o P(x) \tag{3.4}$$

where $\alpha = \frac{R}{X\psi_o}$, and

$$P(x) = \left[\int_0^1 g(x,z)dz\right]^{-1}.$$

Since α is non-zero, the solution to equation (3.4) is

$$f(x) = C_1 \exp\left[\alpha \int_0^x P(\mu)d\mu\right] - \frac{C_o}{\alpha}$$

where C_1 is a constant. Hence

$$f'(x) = C_1 \alpha P(x) \exp\left[\alpha \int_0^x P(\mu)d\mu\right]. \tag{3.5}$$

Since $g(x,z)$ is positive and finite, $P(x)$ is positive and finite. Hence $f'(x)$ as given by (3.5) cannot satisfy either of the boundary conditions (2.8) and (2.9), i.e. $f'(x)$ cannot tend to zero as $x \to \pm\infty$. Thus we have obtained a contradiction and must conclude that the four conditions stated in section 2 cannot all be satisfied for case 1.

Case 2.

For this case X and Z are both of order 1, and if we set $\alpha Z = \lambda$, equation (3.1) can be written as

$$-f'(x) \frac{\partial \phi_o}{\partial z} = X \frac{\partial}{\partial x}(gf') + \lambda \frac{\partial}{\partial z}\left(h \frac{\partial \theta_1}{\partial z}\right).$$

When we integrate over the depth, we get

$$-f'(x)\phi_o(x,z) = X \int_0^z \frac{\partial}{\partial x}(gf')dz + \lambda\left(h \frac{\partial \theta_1}{\partial z}\right) + F_1(x). \tag{3.6}$$

From (2.7) we have that $\frac{\partial \theta_1}{\partial z} = 0$ at $z = 0,1$, and since $h(x,z)$ is finite, equation (3.6) reduces to (3.3). Thus we obtain the same conclusions as for case 1.

Case 3.

In this case $X = O(\alpha)$ and $\alpha Z = O(1)$ and with $\alpha Z = \lambda$, equation (3.1) becomes

$$-f'(x) \frac{\partial \phi_o}{\partial z} = \lambda \frac{\partial}{\partial z}\left(h(x,z) \frac{\partial \theta_1}{\partial z}\right).$$

By integration with respect to z and application of the boundary conditions (2.7) and (2.11) at $z = 0$ we have that

$$f'(x) \equiv 0.$$

This implies that the salinity to the first order is constant, and since we are considering the mixing of fresh and salt water, it must be rejected. Thus we reach the same conclusions as for cases 1 and 2.

In this section and in the next section only one of the conditions $f'(x) \to 0$ as $x \to \pm\infty$ is required. Thus the results of these sections are directly relevant to estuaries.

4. Other Steady Flows

The basic case of steady flow presented above may be extended to several cases of relevance to real estuaries. The first of these arises from the observation that the salinity (or temperature) variation tends to be concentrated in a thin layer to an even greater degree than the velocity variation. Thus a smaller length scale, say D_1, may be appropriate for salinity. Writing $\eta = z/D_1$ the steady form of equation (2.5) now becomes

$$\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial\eta}\frac{D}{D_1} - \frac{\partial\phi}{\partial z}\frac{\partial\theta}{\partial x} = X\frac{\partial}{\partial x}\left(g\frac{\partial\theta}{\partial x}\right) + Z\left(\frac{D}{D_1}\right)^2\frac{\partial}{\partial\eta}\left(h\frac{\partial\theta}{\partial\eta}\right) \qquad (4.1)$$

and hence retaining terms of order α^o and those containing $\alpha(D/D_1)$

$$\frac{\partial\phi_o}{\partial x}\frac{\partial\theta_1}{\partial\eta}\frac{D}{D_1}\alpha - \frac{\partial\phi_o}{\partial z}f'(x) = X\frac{\partial}{\partial x}(gf') + Z\left(\frac{D}{D_1}\right)^2\alpha\frac{\partial}{\partial\eta}\left(h\frac{\partial\theta_1}{\partial\eta}\right). \qquad (4.2)$$

The limited data available suggest that for the estuaries being considered $D/D_1 \sim 2$. Thus the restriction that $\alpha(D/D_1)^2 = O(1)$ may be imposed and hence the first term may be neglected but the final term must be retained, a result supported by the data of Pritchard (2).

Let $Z(D/D_1)^2\alpha = \lambda$. Then equation (4.2) becomes

$$-f'\frac{\partial\phi_o}{\partial z} = X\frac{\partial}{\partial x}(gf') + \lambda\frac{\partial}{\partial\eta}\left(h\frac{\partial\theta_1}{\partial\eta}\right).$$

The final term disappears on integration over the depth and equation (3.2) is again obtained, leading to the same conclusions.

The second extension to the basic steady flow analysis is to permit the stream function to change significantly for small changes in salinity.

In writing equation (2.10) it was assumed that the small stratification $\sum_{n=1}^{\infty}\alpha^n\,\theta_n(x,z)$ produced a flow with correspondingly small

changes in the stream function, $\sum_{n=1}^{\infty}\alpha^n\,\phi_n(x,z)$. If instead it is assumed that the change is of first order, equation (2.10) is replaced by

$$\phi(x,z) = \phi_o(x,z) + M\sum_{n=1}^{\infty}\alpha^n\,\phi_n(x,z)$$

where $M\alpha = \beta = O(1)$, then the following term must be added to the left side of equation (3.1): $-\beta.f'\dfrac{\partial\phi_1}{\partial z}$. On integration of this term with respect to z and the application of the boundary conditions (2.7) and (2.11) it vanishes. Equation (3.3) and the conclusions are thus unaltered.

As the final extension to the steady flow case, the case of flow in which the depth varies gradually along the channel will now be considered.

From (2.5) we see that the governing equation for this case is

$$-\frac{\partial\phi}{\partial z}\frac{\partial\theta}{\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial z} = X\frac{\partial}{\partial x}\left(g\frac{\partial\theta}{\partial x}\right) + Z\frac{\partial}{\partial z}\left(h\frac{\partial\theta}{\partial z}\right). \qquad (4.3)$$

If we linearize this equation as before, we have

$$-f'(x)\frac{\partial\phi_o}{\partial z} = X\frac{\partial}{\partial x}(gf'(x)) \qquad (4.4)$$

where it is supposed that X and Z are both O(1). This equation can be written in the form

$$f''(x) = \left[\frac{\sigma}{\chi g} - \frac{g_x}{g}\right] f'(x)$$

where

$$\sigma(x,z) = -\frac{\partial \phi_o}{\partial z},$$

$$g_x = \frac{\partial g}{\partial x}.$$

If we integrate once, we get

$$f'(x) = \frac{f'(0)g(0,z)}{g(x,z)} \exp\left\{\int_0^x \frac{\sigma(x,z)}{\chi g(x,z)}\, dx\right\}. \qquad (4.5)$$

A similar result can be derived in a different way. Suppose that there is no vertical diffusion near $z = 0$. Then at $z = 0$, equation (4.3) reduces to

$$u_x \frac{\partial \theta}{\partial x} = \chi \frac{\partial}{\partial x} \left(g \frac{\partial \theta}{\partial x}\right)$$

since $u_z = \frac{\partial \phi}{\partial x} = 0$ at $z = 0$. This equation can be integrated to give

$$\frac{\partial \theta}{\partial x} = \frac{K}{g(x,0)} \exp\left\{\frac{1}{\chi} \int_0^x \frac{u_x(x,0)}{g(x,0)}\, dx\right\} \qquad (4.6)$$

where

$$K = g(0,0) \left.\frac{\partial \theta}{\partial x}\right|_{x=0}.$$

Two conclusions can be drawn from (4.5) and (4.6). Firstly since $f'(x)$ and $\frac{\partial \theta}{\partial x}$ must tend to zero as $x \to \infty$ the integrals must be negative at some parts of the channel. Now if the reservoir of salt water is

at $x = \infty$, the horizontal surface velocity is positive and hence $\sigma(x,0)$ is positive. Thus if $g(x,0)$ is positive, $f'(x)$ and $\frac{\partial \theta}{\partial x}$ cannot tend to zero. Secondly, if $g(x,z)$ in (4.5) or $g(x,0)$ in (4.6) become negative, $\sigma(x,z)$ and $u_x(x,0)$ must also become negative in such a way that the quantities

$$\frac{\sigma(x,z)}{g(x,z)} \quad \text{and} \quad \frac{u_x(x,0)}{g(x,0)}$$

do not tend to infinity. Otherwise $f'(x)$ and $\frac{\partial \theta}{\partial x}$ would become infinite. These results are clearly unacceptable on physical grounds.

We notice that the expressions (4.5) and (4.6) can only be obtained if the vertical diffusive term can be removed from equation (4.3).

5. Unsteady Flows

The possibility that the conservation equations and the steady boundary conditions might be satisfied by an unsteady flow was suggested by analogy with the jet streams. The jet streams, through their unsteady meandering, transport momentum in a direction perpendicular to their mean motion balancing advective and other processes and satisfying conservation of momentum. A closer analogy is the salt wedge, which has been observed in the laboratory to advance and retreat without any measurable unsteadiness in the inflow or outflow conditions (Riddell, personal communication, 1971) each retreat being marked by a major increase in the downstream transport of salt by the upper layer.

Both time independent and time dependent zero order profiles are considered and in each case the higher order terms and the diffusion coefficients are allowed to vary with time. Considering first the case of steady zero order salinity, f in the expansion (2.5) is taken as a function only of x.

For greater generality we suppose that

$$\alpha T = \mu = O(1), \quad \alpha Z = \lambda = O(1).$$

Then the linearized equation becomes

$$\mu \frac{\partial \theta_1}{\partial t} = f'(x) \frac{\partial \phi_0}{\partial z} + X \frac{\partial}{\partial x} (gf'(x)) + \lambda \frac{\partial}{\partial z} (h \frac{\partial \theta_1}{\partial z}) \tag{5.1}$$

with boundary conditions as in section 2.

If we integrate equation (5.1) with respect to z and impose the boundary conditions at z = 0,1 we obtain

$$\mu \int_0^1 \frac{\partial \theta_1}{\partial t} dz = - \frac{R}{\psi_0} f'(x) + X \int_0^1 \frac{\partial}{\partial x} (gf'(x))dz. \tag{5.2}$$

As before we assume that the integrands are continuous in x, z, t for $-\infty < x < \infty$, $0 \leq z \leq 1$, $0 \leq t < \infty$ so that the order of integration and differentiation may be exchanged. Hence (5.2) becomes

$$\mu \frac{\partial}{\partial t} \int_0^1 \theta_1 dz = - \frac{R}{\psi_0} f'(x) + X \frac{\partial}{\partial x} f'(x) \int_0^1 g(x,z,t)dz.$$

This equation is integrated with respect to x and again the order of integration and differentiation is exchanged. Hence

$$\mu \frac{\partial}{\partial t} \int_{-\infty}^{x} dx \int_0^1 \theta_1 dz = - \frac{R}{\psi_0} f(x) + X f'(x) \int_0^1 g(x,z,t)dz + H(t)$$

where H(t) is an unknown function. By application of the boundary condition at $x = -\infty$ we see that

$$H(t) = \frac{R}{\psi_0} f(-\infty),$$

and since $f'(x) \to 0$ as $x \to \infty$,

$$\mu \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx \int_0^1 \theta_1 dz = \frac{R}{\psi_0} [f(-\infty) - f(\infty)].$$

Thus

$$\int_{-\infty}^{\infty} dx \int_0^1 \theta_1 dz = \frac{R}{\psi_0 \mu} [f(-\infty) - f(\infty)]t + A$$

where A is a constant.

The left hand side is the total amount of salt in the channel due to θ_1, and since $f(-\infty) - f(\infty)$ must be nonzero, the expression indicates that it becomes infinite as $t \to \infty$. Thus we have obtained the required contradiction.

Now if $T = O(1)$ equation (5.1) becomes

$$0 = f'(x)\frac{\partial\phi_o}{\partial z} + X\frac{\partial}{\partial x}(gf'(x)) + \lambda\frac{\partial}{\partial z}(h\frac{\partial\theta_1}{\partial z}).$$

This case has been treated in section 3 where it was shown that the four conditions cannot all be satisfied.

In the second case of unsteady flow to be considered the zero order salinity is permitted to be unsteady.

If we set $\alpha Z = \lambda = O(1)$ we obtain the following linearized equation

$$T\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial\phi_o}{\partial z} + X\frac{\partial}{\partial x}(g\frac{\partial f}{\partial x}) + \lambda\frac{\partial}{\partial z}(h\frac{\partial\theta_1}{\partial z}).$$

As above we integrate with respect to z and impose the boundary conditions at $z = 0,1$. Thus

$$T\frac{\partial f}{\partial t} = -\frac{R}{\psi_o}\frac{\partial f}{\partial x} + X\int_0^1\frac{\partial}{\partial x}(g\frac{\partial f}{\partial x})dz .$$

Again we suppose that the order of integration and differentiation can be exchanged. After integration with respect to x we obtain

$$T\int_{-\infty}^{x}\frac{\partial f}{\partial t}dx = -\frac{R}{\psi_o}f(x,t) + X\frac{\partial f}{\partial x}\int_0^1 g\,dz + F(t)$$

where $F(t)$ is an unknown function of t. Since

$$\frac{\partial f}{\partial x} \to 0 \quad \text{as } x \to -\infty,$$

$$F(t) = \frac{R}{\psi_o}f(-\infty,t).$$

Thus we have

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}f(x,t)dx = \frac{R}{T\psi_o}\left[f(-\infty,t) - f(\infty,t)\right],$$

since

$$\frac{\partial f}{\partial x} \to 0 \quad \text{as } x \to \infty.$$

Therefore

$$\int_{-\infty}^{\infty}f(x,t)dx = \beta_1 t + \beta_o$$

where β_o = constant, and

$$\beta_1 = \frac{R}{T\psi_o}\left[f(-\infty,t) - f(\infty,t)\right].$$

β_1 is also constant since we have supposed that the salinity and stream-function do not depend on t at the ends of the channel.

This result shows that the salt in the channel due to $f(x,t)$ is unbounded, and thus we have obtained the required contradiction.

6. Discussion of Results

The result obtained is that from consideration of the kinematics and salt conservation a flow cannot be two-dimensional, slightly stratified, with turbulent mixing described by eddy diffusion coefficients, steady and of constant depth; and the same result has been shown to hold with the last two restrictions removed separately. This result has consequences in the formulation of mathematical models of estuarine and other flows which will now be considered.

Estuarine flows, other than sharply stratified cases, have been mathematically modelled one-dimensionally by many workers, but two-dimensional studies are all but non-existent. For this reason alone the analyses of Hansen (3) and Hansen and Rattray (4) are of great interest. Hansen considered an estuarine flow satisfying the above conditions, as did Hansen and Rattray except that for their solution for the inner region they stipulated an exponential or power law variation of the breadth and depth with x_1. To solve the equations they were obliged to impose additional conditions which restricted the validity of their solutions to particular reaches of the estuary. It would be expected that these restricted solutions could be matched at their ends to give a composite solution for the whole estuary. However, the result obtained here shows that the assumptions underlying each of their solutions obtained with constant depth and breadth cannot be applied to the whole estuary so that a composite solution appears to be unobtainable. Hence any improvement upon the pioneering work of Hansen and Rattray, or any newly derived

mathematical models must be based upon some other set of conditions.

It is tempting to speculate on why such an apparently reasonable, if idealized, model should lead to contradictions and with the hope of improving this model each major assumption will be examined in turn. Firstly the assumption of two-dimensional flow, although possibly questionable in a real estuary, can be expected to occur in a laboratory channel. Secondly, the use of eddy diffusion coefficients appears reasonable here, although it does not rest on a sound theoretical basis. This is because no assumptions were made about the form of the coefficients other than to assume that they are greater than zero, i.e. that turbulence would carry more salt from a region of high salinity to one of low salinity than in the opposite direction. The conditions of steady flow and constant depth were relaxed separately and so are unlikely to be responsible for the contradiction.

This leaves the assumption of slight stratification throughout the interior of the estuary and at one end boundary as the most likely fault in the model. Field and laboratory studies now in progress should resolve this point, but whatever the fault it is clear that a rigorous derivation of a mathematical model of estuarine flow is needed and this will form the subject of a later paper in this series.

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